Calculating Galois groups of differential equations with parameters and hypertranscendence

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Abstract

The main motivation of our work is to create an efficient algorithm that decides hypertranscendence of solutions of linear differential equations, via the parameterized differential and Galois theories. To achieve this, we expand the representation theory of linear differential algebraic groups and develop new algorithms that calculate unipotent radicals of parameterized differential Galois groups for differential equations whose coefficients are rational functions.

P. Berman and M. F. Singer presented an algorithm calculating the Galois group for differential equations without parameters whose differential operator is a composition of two completely reducible differential operators. We use their algorithm as a part of our algorithm. As a result, we find an effective criterion for the algebraic independence of the solutions of parameterized differential equations and all of their derivatives with respect to the parameter.

1 Introduction

A special function is called hypertranscendental if it does not satisfy any algebraic differential equations. The study of functional hypertranscendence has recently appeared in various areas of mathematics. In combinatorics, the question of the hypertranscendence of generating series is frequent because it gives information on the growth of the coefficients: for instance, the work of Kurkova and Raschel [27] solved a famous conjecture about the differential algebraic behaviour of generating series of walks on the plane. Dreyfus, Roques, and Hardouin [16] gave criteria to test the hypertranscendence of generating series associated to p-automatic sequences and more generally Mahler functions, generalizing the works of Nguyen [36], Nishioka [37], and Randé [41]. Also, when the derivation encodes the continuous deformation of an auxiliary parameter, the hypertranscendence is connected to the notion of isomonodromic deformations (see the work of Mitschi and Singer [33]).
The works of Cassidy, Hardouin, and Singer [11, 20] were motivated by a study of hypertranscendence using Galois theory. Starting from a linear functional equation with coefficients in a field with a “parametric” derivation, they were able to construct a geometric object, called the parameterized Galois group, whose symmetries control the algebraic relations between the solutions of the functional equation and all of their derivatives. The question of hypertranscendence of solutions of linear functional equations is thus reduced to the computation of the parameterized Galois groups of the equations (see for instance the work of Arreche [1] on the incomplete gamma function $\gamma(x, t)$ and the work [16]). The parameterized Galois groups are linear differential algebraic groups as introduced by Kolchin and developed by Cassidy [6]. These are groups of matrices whose coefficients satisfy systems of polynomial differential equations, called defining equations of the parameterized Galois group.

Then, in this context of Galois theory, one can address a direct problem, that is, the question of the algorithmic computation of the parameterized Galois group. For linear functional equations of order 2, one can find an algorithm initiated by Dreyfus [15] and completed by Arreche [2]. In [32], Minchenko, Ovchinnikov, and Singer gave an algorithm that allows to test if the parameterized Galois group is reductive and to compute the group in that case. In [31], they also show how to compute the parameterized Galois group if its quotient by the unipotent radical is conjugate to a group of matrices with constant entries with respect to the parametric derivations. The algorithms of [31, 32] rely on bounds on the order of the defining equations of the parameterized Galois group, which allows to use the algorithm obtained by Hrushovski [21] and has been further analyzed and improved by Feng [17] in the case of no parametric derivations.

In this paper, we study the parameterized Galois group of a differential operator of the form $L_1(L_2(y)) = 0$ where $L_1, L_2$ are completely reducible differential operators. This situation goes beyond the previously studied cases, because the parameterized Galois group of such an equation is no longer reductive and its quotient by its unipotent radical might not be constant. If there is no parametric derivation, this problem was solved by Berman and Singer in [4] for differential operators and rephrased using Tannakian categories by Hardouin [19]. The general case is however more complicated because, unlike the case of no parameters, the order of the defining equations of the parameterized Galois group is no longer controlled by the order of the functional equation $L_1(L_2(y)) = 0$. Therefore, we present an algorithm that relies on bounds (see §3.3.3) and, in a generic situation, we find a presentation of the parameterized Galois group with defining equations of the unipotent radical expressed by applying standard operations to linear differential operators (cf. [19]).

However, by a careful study of the extension of representations of quasi-simple linear differential algebraic groups, we are able to deduce from our presentation a complete and effective criteria to test the hypertranscendence of solutions of inhomogeneous linear differential equations (Theorem 4.4).

The paper is organized as follows. We start with a brief review of the basic notions in differential algebra, linear differential algebraic groups, and linear differential equations with parameters in Section 2. Our algorithmic results for calculating parameterized Galois groups are in Section 3. Our effective criterion for hypertranscendence of solutions of extensions of irreducible differential equations is in Section 4.2, which is preceded by Section 4.1, where we extend results of Minchenko and Ovchinnikov [30] for the purposes of the hypertranscendence criterion. We use this criterion to analyze solutions of the Lommel differential equation in Section 4.3.

2 Preliminary notions

We shall start with some basic notions of differential algebra and then recall what linear differential algebraic groups and their representations are.
2.1 Differential algebra

**Definition 2.1.** A differential ring is a ring $R$ with a finite set $\Delta$ of commuting derivations on $R$. A $\Delta$-ideal of $R$ is an ideal of $R$ stable under any derivation in $\Delta$.

In the present paper, $\Delta$ will consist of one or two elements. If $R$ is a field and a differential ring, then it is called a differential field, or $\Delta$-field for short. For example, $R = \mathbb{Q}(x, t)$, $\Delta = \{ \partial, \delta \}$, and $\partial = \partial/\partial x$, $\delta = \partial/\partial t$, forms a differential field. The notion of $R$-$\Delta$-algebra is defined analogously.

The ring of $\Delta$-differential polynomials $K[y_1, \ldots, y_n]$ in the differential indeterminates, or $\Delta$-indeterminates, $y_1, \ldots, y_n$ and with coefficients in a $\Delta$-field $(K, \Delta)$, is the ring of polynomials in the indeterminates $\{ \partial y_i | \partial \in \Theta, 1 \leq i \leq n \}$ with coefficients in $K$, where $\Theta = \{ \delta_1^{i_1} \cdots \delta_m^{i_m} | i_1, \ldots, i_m \geq 0 \}$ and $\Delta = \{ \delta_1, \ldots, \delta_m \}$.

**Definition 2.2.** A differential field $(K, \Delta)$ is called differentially closed or $\Delta$-closed for short, if, for every (finite) set of $\Delta$-polynomials $F \subset K[y_1, \ldots, y_n]$, if the system of differential equations $F = 0$ has a solution in some $\Delta$-field extension $L$, then it has a solution with entries in $K$.

For $\partial \in \Delta$, the ring $K[\partial]$ of differential operators, or $\partial$-operators for short, is the $K$-vector space with basis $1, \partial, \ldots, \partial^n, \ldots$ endowed with the following multiplication rule: $\partial \cdot a = a \cdot \partial + \partial(a)$. To a $\partial$-operator $L$ as above, one can associate the linear homogeneous $\partial$-polynomial $L(y) := a_n \partial^n y + \ldots + a_1 \partial y + a_0 y \in K[y]$. In what follows, we assume that every field is of characteristic zero.

2.2 Linear differential algebraic groups and their unipotent radicals

In this section, we first introduce the basic terminology of Kolchin-closed sets, linear differential algebraic groups and their representations. Then we define unipotent radicals of linear differential algebraic groups and their structural properties. We continue with the notion of conjugation to constants of linear differential algebraic groups.

Let $(k, \delta)$ be a differentially closed field and $C$ be the field of $\delta$-constants of $k$. Let $(F, \delta)$ be a $\delta$-subfield of $k$.

2.2.1 First definitions

**Definition 2.3.** A Kolchin-closed (or $\delta$-closed, for short) set $W \subset k^n$ is the set of common zeroes of a system of $\delta$-polynomials with coefficients in $k$, that is, if there exist $f_1, \ldots, f_l \in k[y_1, \ldots, y_n]$ such that

$$W = \{ a \in k^n | f_1(a) = \ldots = f_l(a) = 0 \}.$$

We say that $W$ is defined over $F$ if $W$ is the set of zeroes of $\delta$-polynomials with coefficients in $F$. More generally, for a $\delta$-$F$-algebra $R$,

$$W(R) = \{ a \in R^n | f_1(a) = \ldots = f_l(a) = 0 \}.$$

**Definition 2.4.** If $W \subset k^n$ is a Kolchin-closed set defined over $F$, the $\delta$-ideal

$$I(W) = \{ f \in F[y_1, \ldots, y_n] | f(w) = 0 \text{ for all } w \in W(k) \}$$

is called the defining $\delta$-ideal of $W$ over $F$. Conversely, for a subset $S$ of $F[y_1, \ldots, y_n]$, the following subset is $\delta$-closed in $k^n$:

$$V(S) := \{ a \in k^n | f(a) = 0 \text{ for all } f \in S \}.$$
Definition 2.5. Let $W \subset \mathbf{k}^n$ be a $\delta$-closed set defined over $F$. The $\delta$-coordinate ring $F[W]$ of $W$ over $F$ is the $F$-$\Delta$-algebra

$$F[W] = F[y_1, \ldots, y_n]/I(W).$$

If $F[W]$ is an integral domain, then $W$ is called irreducible. This is equivalent to $I(W)$ being a prime $\delta$-ideal.

Example 2.6. The affine space $\mathbf{A}^n$ is the irreducible Kolchin-closed set $\mathbf{k}^n$. It is defined over $F$, and its $\delta$-coordinate ring over $F$ is $F[y_1, \ldots, y_n]$.

Definition 2.7. Let $W \subset \mathbf{k}^n$ be a $\delta$-closed set defined over $F$. Let $I(W) = p_1 \cap \ldots \cap p_q$ be a minimal $\delta$-prime decomposition of $I(W)$, i.e., the $p_i \subset F[y_1, \ldots, y_n]$ are prime $\delta$-ideals containing $I(W)$ and minimal with this property. This decomposition is unique up to permutation ([24, VII.29]). Then the irreducible Kolchin-closed sets $W_i = V(p_i)$ are defined over $F$ and called the irreducible components of $W$. We have $W = W_1 \cup \ldots \cup W_q$.

Definition 2.8. Let $W_1 \subset \mathbf{k}^{n_1}$ and $W_2 \subset \mathbf{k}^{n_2}$ be two Kolchin-closed sets defined over $F$. A $\delta$-polynomial map (morphism) defined over $F$ is a map

$$\varphi : W_1 \to W_2, \quad a \mapsto (f_1(a), \ldots, f_{n_2}(a)),$$

where $f_i \in F[y_1, \ldots, y_{n_i}]$ for all $i = 1, \ldots, n_2$.

If $W_1 \subset W_2$, the injection of $W_1$ in $W_2$ is a $\delta$-polynomial map. In this case, we say that $W_1$ is a $\delta$-closed subset of $W_2$.

Example 2.9. Let $\text{GL}_n \subset \mathbf{k}^n$ be the group of $n \times n$ invertible matrices with entries in $\mathbf{k}$. One can see $\text{GL}_n$ as a Kolchin-closed subset of $\mathbf{k}^{n^2} \times \mathbf{k}$ defined over $F$, via the embedding

$$\text{GL}_n \to \mathbf{k}^{n^2} \times \mathbf{k}, \quad A \mapsto (A, 1/\det(A)).$$

The $\delta$-coordinate ring of $\text{GL}_n$ over $F$ is $F[Y, 1/\det(Y)]$, where $Y = (y_{i,j})_{1 \leq i, j \leq n}$ is a matrix of $\delta$-indeterminates over $F$. We also denote by $\text{SL}_n \subset \text{GL}_n$ the special linear group that consists of the matrices of determinant 1.

Similarly if $V$ is a finite-dimensional $F$-vector space, $\text{GL}(V)$ is defined as the group of invertible $\mathbf{k}$-linear maps of $V \otimes_F \mathbf{k}$. To simplify the terminology, we will also treat $\text{GL}(V)$ as Kolchin-closed sets tacitly assuming that some basis of $V$ is fixed.

Remark 2.10. If $K$ is a field, we denote by $\text{GL}_n(K)$ the group of invertible matrices with coefficients in $K$.

Definition 2.11 ([6, Chapter II, Section 1, p. 905]). A linear differential algebraic group $G \subset \mathbf{k}^{n_2}$ defined over $F$ is a subgroup of $\text{GL}_n$, that is a Kolchin-closed set defined over $F$. If $G \subset H \subset \text{GL}_n$ are Kolchin-closed subgroups of $\text{GL}_n$, we say that $G$ is a $\delta$-closed subgroup, or $\delta$-subgroup of $H$.

Proposition 2.12. Let $G \subset \text{GL}_n$ be a linear algebraic group defined over $F$. Then, $G$ is a linear differential algebraic group.

Let $H \subset G$ be a $\delta$-subgroup of $G$ defined over $F$. The Zariski closure $\overline{H} \subset G$ is the closure of $H$ with respect to the Zariski topology. Then, $\overline{H}$ is a linear algebraic group defined over $F$, whose polynomial defining ideal over $F$ is $I(H) \cap F[Y] \subset I(H) \subset F[Y]$, where $Y = (y_{i,j})_{1 \leq i, j \leq n}$ is a matrix of $\delta$-indeterminates over $F$.

Definition 2.13. Let $G$ be a linear differential algebraic group defined over $F$. The irreducible component of $G$ containing $e$, the identity element, is called the identity component of $G$ and denoted by $G^0$. The linear differential algebraic group $G^0$ is a $\delta$-subgroup of $G$ defined over $F$. The linear differential algebraic group $G$ is called connected if $G = G^0$, which is equivalent to $G$ being an irreducible Kolchin-closed set [6, p. 906].
Definition 2.14 ([7],[39, Definition 6]). Let $G$ be a linear differential algebraic group defined over $F$ and let $V$ be a finite-dimensional vector space over $F$. A $\delta$-polynomial group homomorphism $\rho : G \to \text{GL}(V)$ defined over $F$ is called a representation of $G$ over $F$. We shall also say that $V$ is a $G$-module over $F$. By faithful (respectively, semisimple) $G$-module, we mean faithful (respectively, completely reducible) representation $\rho : G \to \text{GL}(V)$.

The image of a $\delta$-polynomial group homomorphism $\rho : G \to H$ is Kolchin closed [6, Proposition 7]. Moreover, if $\ker(\rho) = \{e\}$, then $\rho$ is an isomorphism of linear differential algebraic groups $G$ and $\rho(G)$ [6, Proposition 8].

Definition 2.15 ([8, Theorem 2]). A linear differential algebraic group $G$ is unipotent if one of the following equivalent conditions holds:

1. $G$ is conjugate to a differential algebraic subgroup of the group of unipotent upper triangular matrices;
2. $G$ contains no elements of finite order $> 1$;
3. $G$ has a descending normal sequence of differential algebraic subgroups

$$G = G_0 \supset G_1 \supset \ldots \supset G_N = \{e\}$$

with $G_i / G_{i+1}$ isomorphic to a differential algebraic subgroup of the additive group $\text{Ga}$.

One can show that a linear differential algebraic group $G$ defined over $F$ admits a largest normal unipotent differential algebraic subgroup defined over $F$ [29, Theorem 3.10].

Definition 2.16. Let $G$ be a linear differential algebraic group defined over $F$. The largest normal unipotent differential algebraic subgroup of $G$ defined over $F$ is called the unipotent radical of $G$ and denoted by $R_u(G)$. The unipotent radical of a linear algebraic group $H$ is also denoted by $R_u(H)$.

Note that, for a linear differential algebraic group $G$, we always have $R_u(G) \subset R_u(G)$ and this inclusion can be strict [29, Example 3.17].

2.2.2 Almost direct products and reductive linear differential algebraic group

We recall what reductive linear differential algebraic groups are and how they decompose into almost direct products of tori and quasi-simple subgroups.

Definition 2.17. A linear differential algebraic group $G$ is called simple if $\{e\}$ and $G$ are the only normal differential algebraic subgroups of $G$.

Definition 2.18. A quasi-simple linear (differential) algebraic group is a finite central extension of a simple non-commutative linear (differential) algebraic group.

Definition 2.19 ([29, Definition 3.12]). A linear differential algebraic group $G$ defined over $F$ is called reductive if $R_u(G) = \{e\}$.

By definition, the following holds for linear differential algebraic groups:

simple $\implies$ quasi-simple $\implies$ reductive.

Example 2.20. $\text{SL}_2$ is quasi-simple but not simple, while $\text{PSL}_2$ is simple.

Proposition 2.21 ([32, Remark 2.9]). Let $G \subset \text{GL}_n$ be a linear differential algebraic group defined over $F$. If $\overline{G} \subset \text{GL}_n$ is a reductive linear algebraic group, then $G$ is a reductive linear differential algebraic group.
**Definition 2.22.** Let $G$ be a group and $G_1, \ldots, G_n$ be some subgroups of $G$. We say that $G$ is the almost direct product of $G_1, \ldots, G_n$ if

1. the commutator subgroups $[G_i, G_j] = \{e\}$ for all $i \neq j$;
2. the morphism $\psi : G_1 \times \cdots \times G_n \rightarrow G, (g_1, \ldots, g_n) \mapsto g_1 \cdot \cdots \cdot g_n$ is an isogeny, that is, a surjective map with finite kernel.

We summarize some results on the decomposition of reductive, algebraic and differential algebraic, groups in the theorem below. We refer to Definition 2.3 for the notation $G(C)$ with $G$ a linear (differential) algebraic group defined over $C$.

**Theorem 2.23.** Let $G \subset \text{GL}_n$ be a linear differential algebraic group defined over $F$. Assume that $\overline{G} \subset \text{GL}_n$ is a connected reductive algebraic group. Then

1. $\overline{G}$ is an almost direct product of a torus $T$ and non-commutative normal quasi-simple linear algebraic groups $G_1, \ldots, G_s$;
2. $G$ is an almost direct product of a Zariski dense $\delta$-closed subgroup $Z$ of $T$ and some $\delta$-closed subgroups $H_i$ of $G_i$ for $i = 1, \ldots, s$;
3. moreover, either $G_i = H_i$ or $H_i$ is conjugate by a matrix of $G_i$ to $G_i(C)$;

The $G_i$’s are called the quasi-simple components of $G$; the $H_i$’s are called the $\delta$-quasi-simple components of $G$.

**Proof.** Part (1) can be found in [22, Theorem 27.5, page 167]. Parts (2) and (3) are contained in [29, proof of Lemma 4.5] and [9, Theorems 15 and 18].

**Remark 2.24.** As noticed in [32, §5.3.1], the decomposition of $\overline{G}$ as above can be made effective.

### 2.2.3 Conjugation to constants

Conjugation to constants will play an essential role in our arguments. We now recall what it is. Recall that $k$ is a differentially closed field containing $F$ and $C$ is the field of $\delta$-constants of $k$.

**Definition 2.25.** Let $G \subset \text{GL}_n$ be a linear algebraic group over $F$. We say that $G$ is conjugate to constants if there exists $h \in \text{GL}_n$ such that $hGh^{-1} \subset \text{GL}_n(C)$. Similarly, we say that a representation $\rho : G \rightarrow \text{GL}_n$ is conjugate to constants if $\rho(G)$ is conjugate to constants in $\text{GL}_n$.

**Proposition 2.26.** Let $\rho : G \subset \text{GL}(W) \rightarrow \text{GL}(V)$ be a representation of a linear differential algebraic group $G$ with $\overline{G} \subset \text{GL}(W)$ being a connected reductive linear algebraic group. Assume that $\rho$ is defined over the field $C$. With the notation of Theorem 2.23, assume that $Z$ acts by constant weights on $V$ and that, for all $i = 1, \ldots, s$, either $H_i \neq G_i$ or $\rho|_{H_i}$ is trivial. Then there exists $g \in \overline{G}$ such that $\rho(gGg^{-1}) \subset \text{GL}(V)(C)$.

**Proof.** Let $S = \{i \mid H_i = G_i\}$. By assumption, $\rho(H_i) = \{1\}$ for all $i \in S$. By Theorem 2.23, for all $i \notin S$, there exists $g_i \in G_i$ such that $g_iH_ig_i^{-1} \subset G_i(C)$. Set

$$g := \prod_{i \in S} g_i \in G.$$ 

Now, let $h \in G$. Since $G$ is the almost direct product of $Z$ and of its $\delta$-quasi-simple components, there exist $z \in Z$ and, for $i \in \{1, \ldots, s\}$, an element $h_i \in H_i$ such that $h = zh_1 \cdot \cdots \cdot h_s$. Now,

$$\rho(ghg^{-1}) = \rho(z) \prod_{i \in S} \rho(g_ih_ig_i^{-1}).$$

Since $\rho$ is defined over the constants and $g_ih_ig_i^{-1} \subset G_i(C)$ for all $i \notin S$, we find that $\rho(g_ih_ig_i^{-1}) \subset \text{GL}(V)(C)$. Since $\rho(z)$ is also constant, the same holds for $\rho(ghg^{-1})$. □
2.3 Parameterized differential modules

In this section, we recall the basic definitions of differential modules and prolongation functors for differential modules with parameters. We then continue with the notion of complete integrability of differential modules and its relation to conjugation to constants of parameterized Galois groups. We also show a new result, Proposition 2.48, which relates the conjugation to constants of a linear differential algebraic group and of its identity component.

2.3.1 Differential modules and prolongations

Let $K$ be a $\Delta = \{\partial, \delta\}$-field. We denote by $k$ (respectively, $C$) the field of $\partial$ (respectively, $\Delta$)-constants of $K$. We assume for simplicity that $(k, \delta)$ is differentially closed (this assumption was relaxed in [18, 47, 35]). Therefore, unless explicitly mentioned, any Kolchin-closed set considered in the rest of the paper is a subset of some $k^n$.

**Definition 2.27.** A $\partial$-module $\mathcal{M}$ over $K$ is a left $K[\partial]$-module that is a finite-dimensional vector space over $K$.

Let $\mathcal{M}$ be a $\partial$-module over $K$ and let $\{e_1, \ldots, e_n\}$ be a $K$-basis of $\mathcal{M}$. Let $A = (a_{i,j}) \in K^{n \times n}$ be the matrix defined by

$$\partial(e_i) = -\sum_{j=1}^{n} a_{j,i} e_j, \quad i = 1, \ldots, n. \quad (2.1)$$

Then, for any element $m = \sum_{i=1}^{n} y_i e_i$, where $Y = (y_1, \ldots, y_n)^T \in K^n$, we have

$$\partial(m) = \sum_{i=1}^{n} \partial(y_i) e_i - \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{i,j} y_j \right) e_i.$$

Thus, the equation $\partial(m) = 0$ translates into the linear differential system $\partial(Y) = AY$.

**Definition 2.28.** Let $\mathcal{M}$ be a $\partial$-module over $K$ and $\{e_1, \ldots, e_n\}$ be a $K$-basis of $\mathcal{M}$. We say that the linear differential system $\partial(Y) = AY$, as above, is associated to the $\partial$-module $\mathcal{M}$ (via the choice of a $K$-basis). Conversely, given a linear differential system $\partial(Y) = AY$, $A = (a_{i,j}) \in K^{n \times n}$, one associates a $\partial$-module $\mathcal{M}$ over $K$, namely: $\mathcal{M} = K^n$ with the standard basis $(e_1, \ldots, e_n)$ and action of $\partial$ given by (2.1).

Another choice of a $K$-basis $X := BY$, where $B \in GL_n(K)$, leads to the differential system $\partial(X) = (B^{-1} AB - B^{-1} \partial(B))X$.

**Definition 2.29.** We say that a linear differential system $\partial(X) = \bar{A}X$, with $\bar{A} \in K^{n \times n}$, is gauge equivalent to a linear differential system $\partial(X) = AX$, with $A \in K^{n \times n}$, if there exists $B \in GL_n(K)$ such that $\bar{A} = B^{-1} AB - B^{-1} \partial(B)$.

One has the following correspondence between linear differential systems and linear differential equations. For $L = \partial^n + a_{n-1} \partial^{n-1} + \ldots + a_0 \in K[\partial]$, one can consider the companion matrix

$$A_L := \begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & \ddots &  & \vdots \\
\vdots & \ddots & 0 & 1 \\
-a_0 & -a_1 & \cdots & -a_{n-1}
\end{pmatrix}.$$

The differential system $\partial Y = A_L Y$ induces a $\partial$-module structure on $K^n$, which we denote by $\mathcal{L}$. Conversely, the cyclic vector lemma [44, Proposition 2.9] states that any $\partial$-module is isomorphic to a $\partial$-module $\mathcal{L}$, of the above form, provided $k \subseteq K$. 7
Definition 2.30. A morphism of $\partial$-modules over $K$ is a homomorphism of $K[\partial]$-modules.

One can, therefore, consider the category $\text{Diff}_K$ of $\partial$-modules over $K$. This category has a rich structure, and one can define several operations on its objects as follows.

Definition 2.31. We can define the following constructions in $\text{Diff}_K$:

1. The direct sum of two $\partial$-modules, $\mathcal{M}_1$ and $\mathcal{M}_2$, is $\mathcal{M}_1 \oplus \mathcal{M}_2$ together with the action of $\partial$ defined by $\partial(m_1 + m_2) = \partial(m_1) + \partial(m_2)$.

2. The tensor product of two $\partial$-modules, $\mathcal{M}_1$ and $\mathcal{M}_2$, is $\mathcal{M}_1 \otimes_K \mathcal{M}_2$ together with the action of $\partial$ defined by $\partial(m_1 \otimes m_2) = \partial(m_1) \otimes m_2 + m_1 \otimes \partial(m_2)$.

3. The unit object $1$ for the tensor product is the field $K$ together with the left $K[\partial]$-module structure given by $(a_0 + a_1 \partial + \cdots + a_n \partial^n)(f) := a_0 f + \cdots + a_n \partial^n(f)$ for $f, a_0, \ldots, a_n \in K$.

4. The internal Hom of two $\partial$-modules $\mathcal{M}_1, \mathcal{M}_2$ exists in $\text{Diff}_K$ and is denoted by $\text{Hom}(\mathcal{M}_1, \mathcal{M}_2)$. It consists of the $K$-vector space $\text{Hom}_K(\mathcal{M}_1, \mathcal{M}_2)$ of $K$-linear maps from $\mathcal{M}_1$ to $\mathcal{M}_2$ together with the action of $\partial$ given by the formula $\partial(m_1) = \partial(u(m_1)) - u(\partial m_1)$. The dual $\mathcal{M}^*$ of a $\partial$-module $\mathcal{M}$ is the $\partial$-module $\text{Hom}(\mathcal{M}, 1)$.

5. An endofunctor $D : \text{Diff}_K \to \text{Diff}_K$, called the prolongation functor, is defined as follows: if $\mathcal{M}$ is an object of $\text{Diff}_K$ corresponding to the linear differential system $\partial(Y) = AY$, then $D(\mathcal{M})$ corresponds to the linear differential system $\partial(Z) = \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix} Z$.

The construction of the prolongation functor reflects the following idea. If $U$ is a fundamental solution matrix of $\partial(Y) = AY$ in some $\Delta$-field extension $F$ of $K$, that is, $\partial(U) = AU$ and $U \in \text{GL}_n(F)$, then

$$\partial(U) = \delta(U) = \delta(A)U + A\delta(U).$$

Then, $\begin{pmatrix} U & \delta(U) \\ 0 & U \end{pmatrix}$ is a fundamental solution matrix of $\partial(Z) = \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix} Z$. Endowed with all these constructions, the category $\text{Diff}_K$ is a $\partial$-tensor category in the sense of [40, 23].

Theorem 2.32. The category $(\text{Diff}_K, \otimes, 1, D)$ is a $\partial$-tensor category over $k$, that is, a rigid abelian tensor category such that $k = \text{End}(1)$, enriched with an endofunctor $D$, called the prolongation functor, satisfying the conditions in [40, Definition 3]. If we forget the functor $D$, we find that the category $(\text{Diff}_K, \otimes, 1)$ is a tensor category in the sense of [12].

In this paper, we will not consider the whole category $\text{Diff}_K$ but the $\partial$-tensor subcategory generated by a $\partial$-module. More precisely, we have the following definition.

Definition 2.33. Let $\mathcal{M}$ be an object of $\text{Diff}_K$. We denote by $\{\mathcal{M}\}^\otimes,\partial$ the smallest full subcategory of $\text{Diff}_K$ that contains $\mathcal{M}$ and is closed under all operations of linear algebra (direct sums, tensor products, duals, and subquotients) and under $D$. The category $\{\mathcal{M}\}^\otimes,\partial$ is a $\partial$-tensor category over $k$. We also denote by $\{\mathcal{M}\}^\otimes$ the full tensor subcategory of $\text{Diff}_K$ generated by $\mathcal{M}$. Then, $\{\mathcal{M}\}^\otimes$ is a tensor category over $k$.

Similarly, the category $\text{Vect}_k$ of finite-dimensional $k$-vector spaces is a $\partial$-tensor category. The prolongation functor on $\text{Vect}_k$ is defined as follows: for a $k$-vector space $V$, the $k$-vector space $D(V)$ equals $k[\delta]_{\leq 1} \otimes_k V$, where $k[\delta]_{\leq 1}$ is considered as the right $k$-module of $\partial$-operators up to order 1 and $V$ is viewed as a left $k$-module.

Definition 2.34. Let $\mathcal{M}$ be an object of $\text{Diff}_K$. A $\partial$-fiber functor $\omega : \{\mathcal{M}\}^\otimes,\partial \to \text{Vect}_k$ is an exact, faithful, $k$-linear, tensor compatible functor together with a natural isomorphism between $D_{\text{Vect}_k} \circ \omega$ and $\omega \circ D_{\{\mathcal{M}\}^\otimes,\partial}$ [23, Definition 4.2.7], where the subscripts emphasize the category on which we do the prolongation. The pair $\{\mathcal{M}\}^\otimes,\partial, \omega$ is called a $\partial$-Tannakian category.
Remark 2.38. The parameterized Galois group depends a priori on the choice of a \( \omega \)-dimensional representations of \( \text{Gal}(\omega) \) that stabilize \( \omega(\mathcal{V}) \) for every \( \omega \)-module \( \mathcal{V} \) obtained from \( \mathcal{M} \) by applying the linear constructions (subquotient, direct sum, tensor product, and dual), and the prolongation functor. The action of \( g \) on \( \omega(\mathcal{V}) \) is obtained by applying the same constructions to \( g \). We call \( \text{Gal}^\delta(\mathcal{M}) \) the parameterized Galois group of \( (\mathcal{M}, \omega) \), or of \( \mathcal{M} \) when there is no confusion.

Theorem 2.37 ([40, Theorem 2]). Let \( \mathcal{M} \) be an object of \( \text{Diff}_k \) and \( \omega : (\mathcal{M})^{\otimes,\delta} \to \text{Vect}_k \) be a \( \delta \)-fiber functor. The group \( \text{Gal}^\delta(\mathcal{M}) \subset \text{GL}(\omega(\mathcal{M})) \) is a linear differential algebraic group defined over \( k \), and \( \omega \) induces an equivalence of categories between \( (\mathcal{M})^{\otimes,\delta} \) and the category of \( \delta \)-fiber functors and any two \( \delta \)-fiber functors for \( \mathcal{M} \) are isomorphic as linear differential algebraic groups over \( k \).

Remark 2.38. The parameterized Galois group depends a priori on the choice of a \( \delta \)-fiber functor \( \omega \). However, since two \( \delta \)-fiber functors for \( (\mathcal{M})^{\otimes,\delta} \) are naturally isomorphic, we find that the parameterized Galois groups that these functors define are isomorphic as linear differential algebraic groups over \( k \). Thus, if it is not necessary, we will speak of the parameterized Galois group of \( \mathcal{M} \) without mentioning the \( \delta \)-fiber functor.

Forgetting the action of \( \delta \), one can similarly define the group \( \text{Gal}(\mathcal{M}) \) of tensor isomorphisms of \( \omega : (\mathcal{M})^{\otimes} \to \text{Vect}_k \). By [12], the group \( \text{Gal}(\mathcal{M}) \subset \text{GL}(\omega(\mathcal{M})) \) is a linear algebraic group defined over \( k \), and \( \omega \) induces an equivalence of categories between \( (\mathcal{M})^{\otimes} \) and the category of \( k \)-finite-dimensional representations of \( \text{Gal}(\mathcal{M}) \).

Proposition 2.39 ([20, Proposition 6.21]). Let \( \mathcal{M} \) be an object of \( \text{Diff}_K \) and \( \omega : (\mathcal{M})^{\otimes,\delta} \to \text{Vect}_K \) be a \( \delta \)-fiber functor. Then \( \text{Gal}^\delta(\mathcal{M}) \) is a Zariski dense subgroup of \( \text{Gal}(\mathcal{M}) \) (see Proposition 2.12).

Definition 2.40. A parameterized Picard–Vessiot extension, or PPV-extension for short, of \( K \) for a \( \delta \)-module \( \mathcal{M} \) over \( K \) is a \( \delta \)-field extension \( K_{\mathcal{M}} \) that is generated over \( K \) by the entries of a fundamental solution matrix \( U \) of a differential system \( \delta(X) = AX \) associated to \( \mathcal{M} \) and such that \( K_{\mathcal{M}}^\delta = K^\delta \). The field \( K(U) \) is a Picard–Vessiot extension, that is, a \( \delta \)-field extension of \( K \) generated by the entries of a fundamental solution matrix \( U \) of \( \delta(X) = AX \) such that \( K(U)^\delta = K^\delta \).

A parameterized Picard–Vessiot extension associated to a \( \delta \)-module \( \mathcal{M} \) depends a priori on the choice of a \( K \)-basis of \( \mathcal{M} \), which is equivalent to the choice of a linear differential system associated to \( \mathcal{M} \). However, one can show that gauge equivalent differential systems lead to parameterized Picard–Vessiot extensions that are isomorphic as \( K \)-\( \Delta \)-algebras. In [12], Deligne showed that a fiber functor corresponds to a Picard–Vessiot extension; it is shown in [18, Theorem 5.5] that the notions of \( \delta \)-fiber functor and parameterized Picard–Vessiot extension are equivalent.

Definition 2.41. Let \( \mathcal{M} \) be a \( \delta \)-module over \( K \). Let \( \delta(X) = AX \) be a differential system associated to \( \mathcal{M} \) over \( K \) with \( A \in K^{n \times n} \) and let \( K_{\mathcal{M}} \) be a PPV-extension for \( \delta(X) = AX \) over \( K \). The parameterized Picard–Vessiot group, or PPV-group for short, \( \text{Gal}^\delta(K_{\mathcal{M}}/K) \) is the set of \( \Delta \)-automorphisms of \( K_{\mathcal{M}} \) over \( K \) whereas the Picard–Vessiot group is the set of \( \Delta \)-automorphisms of a Picard–Vessiot extension of \( K \).

Remark 2.42. Let \( U \in \text{GL}_n(K_{\mathcal{M}}) \) be a fundamental solution matrix of \( \delta(X) = AX \). For any \( \tau \in \text{Gal}^\delta(K_{\mathcal{M}}/K) \), there exists \( [\tau]_U \in \text{GL}_n(k) \) such that \( \tau(U) = U[\tau]_U \). The map

\[
\text{Gal}^\delta(K_{\mathcal{M}}/K) \to \text{GL}_n, \quad \tau \mapsto [\tau]_U
\]

9
is an embedding and identifies $\text{Gal}^\delta(K_{\mathcal{M}}/K)$ with a $\delta$-closed subgroup of $\text{GL}_n$. One can show that another choice of fundamental solution matrix as well as another choice of gauge equivalent linear differential system yield a conjugate subgroup in $\text{GL}_n$. Similarly, one can represent $\text{Gal}(K_{\mathcal{M}}/K)$ as a linear algebraic subgroup of $\text{GL}_n$. With these representations of the Picard–Vessiot groups, one can show that Picard–Vessiot groups and Galois groups are isomorphic in the parameterized and non parameterized cases.

In the PPV theory, a Galois correspondence holds between differential algebraic subgroups of the PPV-group and $\Delta$-sub-field extensions of $K_{\mathcal{M}}$ (see [20, Theorem 6.20] for more details). Moreover, the $\delta$-dimension of $\text{Gal}^\delta(K_{\mathcal{M}})$ coincides with the $\delta$-transcendence degree of $K_{\mathcal{M}}$ over $K$ (see [20, p. 374 and Proposition 6.26] for the definition of the $\delta$-dimension and $\delta$-transcendence degree and the proof of their equality). Moreover, the defining equations of the parameterized Galois group reflect the differential algebraic relations among the solutions (see [20, Proposition 6.24]). Therefore, given a $\partial$-module $\mathcal{M}$ over $K$, we find that the defining equations of the parameterized Galois group $\text{Gal}^\delta(\mathcal{M})$ over $K$ determine the differential algebraic relations between the solutions in $K_{\mathcal{M}}$ over $K$.

**Definition 2.43.** A $\delta$-module $\mathcal{M}$ is called completely reducible if, for every $\partial$-submodule $\mathcal{N}$ of $\mathcal{M}$, there exists a $\partial$-submodule $\mathcal{N}'$ of $\mathcal{M}$ such that $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}'$. We say that a $\partial$-operator is completely reducible if the associated $\partial$-module is completely reducible.

By [44, Exercise 2.38], a $\partial$-module is completely reducible if and only if its Galois group is a reductive linear algebraic group. Moreover, for a completely reducible $\partial$-module $\mathcal{M}$, any object in $\mathcal{M}^\otimes$ is completely reducible.

### 2.3.2 Isomonodromic differential modules

**Definition 2.44 ([11, Definition 3.8]).** Let $A \in K^{n \times n}$. We say that the linear differential system $\partial Y = AY$ is isomonodromic (or completely integrable) over $K$ if there exists $B \in K^{n \times n}$ such that $\partial(B) - \delta(A) = AB - BA$.

**Remark 2.45.** One can show that a linear differential system $\partial Y = AY$ is isomonodromic if and only if there exists a $\Delta$-field extension $L$ of $K$ and $B \in K^{n \times n}$ such that the system

\[
\begin{align*}
\partial Y &= AY \\
\delta Y &= BY
\end{align*}
\]

has a fundamental solution matrix with coefficients in $L$.

We recall a characterization of complete integrability in terms of the PPV theory.

**Proposition 2.46 ([11, Proposition 3.9]).** Let $\mathcal{M}$ be a $\partial$-module over $K$ and $\partial(Y) = AY$, with $A \in K^{n \times n}$, be an associated linear differential system. The following statements are equivalent:

- $\text{Gal}^\delta(\mathcal{M})$ is conjugate to constants in $\text{GL}(\omega(\mathcal{M}))$ (see Definition 2.25);
- The linear differential system $\partial(Y) = AY$ is isomonodromic over $K$.

The proof of the following result was provided to the authors by Michael F. Singer and will be used in the proof of Proposition 2.48.

**Lemma 2.47.** Given a linear differential algebraic group $G \subset \text{GL}_n$ defined over a differentially closed field $(\mathbb{k}, \delta)$ and any $\Delta = (\partial, \delta)$-field $K$ such that $K^\delta = \mathbb{k}$, there exists a $\Delta$-field extension $F$ of $K$ such that $F^\delta = \mathbb{k}$ and $G$ can be realized as a parameterized Galois group over $F$ in the given faithful representation of $G \subset \text{GL}_n$. 

10
Proof. We first consider the “generic” case. That is, we first construct a \( \Delta \)-field extension \( E \) of \( K \) with no new \( \partial \)-constants such that \( \text{GL}_n \) is a parameterized Galois group of a \( \partial \)-module \( \mathcal{M} \) over \( E \). Assume we have constructed \( E \) and let \( E_{\mathcal{M}} \) be a PPV extension of \( \mathcal{M} \) over \( E \). For any differential algebraic subgroup \( G \) of \( \text{GL}_n \), let \( F \) be the fixed field of \( G \) in \( E_{\mathcal{M}} \), i.e., the elements of \( E_{\mathcal{M}} \) fixed by \( G \). By the PPV correspondence, \( G \) is the parameterized Galois group of \( E_{\mathcal{M}} \) over \( F \). Moreover, \( K^\partial = k \subset F^\partial \subset E^\partial = k \).

To construct the fields \( E_{\mathcal{M}} \) and \( E \) for \( \text{GL}_n \), we shall follow the construction from [28, pages 87–89]. Let \( \{ z_{i,j} \} \) be a set of \( n^2 \) \( \Delta \)-differential indeterminates over \( K \). Let \( E_{\mathcal{M}} = K(\{ z_{i,j} \})_\Delta \) be a \( \Delta \)-field of differential rational functions in these indeterminates. Note that the \( \delta \)-constants of \( E_{\mathcal{M}} \) are \( k \), as in [28, Lemma 2.14]. Let \( Z = (z_{i,j}) \in \text{GL}_n(E_{\mathcal{M}}) \) and \( A = (\partial Z)(Z)^{-1} \). We then have that

\[
\partial Z = AZ. \tag{2.2}
\]

Let \( E \) be the \( \Delta \)-field generated over \( K \) by the entries of \( A \). Then, \( E_{\mathcal{M}} \) is a PPV extension of \( E \) for equation (2.2). Since \( Z \) is a matrix of \( \Delta \)-differential indeterminates, any assignment \( Z \to Zg \) for \( g \in \text{GL}_n(K) \) defines a \( \Delta \)-\( K \)-automorphism \( \phi_g \) of \( E_{\mathcal{M}} \) over \( K \). If we restrict to those \( g \in \text{GL}_n \), then \( \phi_g \) leaves \( A \) fixed and so all elements of \( E \) are left fixed. Therefore, \( \text{GL}_n \) is a subgroup of the PPV-group of \( E_{\mathcal{M}} \) over \( E \). Since this PPV-group is already a subgroup of \( \text{GL}_n \), we must have that the PPV-group of \( E_{\mathcal{M}} \) over \( E \) is \( \text{GL}_n \).

The proof of the following result uses PPV theory, which does not appear in the statement. It is, therefore, of interest to find a direct proof of it as well.

**Proposition 2.48.** Let \( G \subset \text{GL}(V) \) be a linear differential algebraic group over \( k \) and let \( G^\partial \) be the identity component of \( G \). If \( G^\partial \) is conjugate to constants in \( \text{GL}(V) \), then the same holds for \( G \).

**Proof.** By Lemma 2.47, let \( K \) be a \( \Delta \)-field with \( K^\partial = k \) such that \( G \) is a parameterized Galois group of a \( \partial \)-module \( \mathcal{M} \) over \( K \) and the embedding \( G \subset \text{GL}(V) \) is the faithful representation \( G \to \text{GL}(\omega(\mathcal{M})) \). Let \( L/K \) be a PPV extension for \( \mathcal{M} \) over \( K \). One can identify \( G \) with \( \text{Gal}(L/K) \), the group of automorphisms of \( L \) over \( K \) commuting with \( \partial \) and \( \delta \). Let \( F \) be the subfield of \( L \) fixed by \( G^\partial \). By the PPV correspondence [11, Theorem 9.5], the group of automorphisms of \( L \) over \( F \) commuting with \( \{ \delta, \partial \} \) coincides with \( G^\partial \) and the extension \( F/K \) is algebraic since \( G/G^\partial \) is finite.

Let \( \partial(Y) = AY \) be a linear differential system associated to \( \mathcal{M} \). The parameterized Galois group of \( \mathcal{M} \) over \( F \) is \( G^\partial \) and thus conjugate to constants by assumption. Proposition 2.46 implies that \( \partial(Y) = AY \) is isomonodromic over \( F \). That is, there exists \( B \in F^{n \times n} \) such that

\[
\partial(B) - \partial(A) = AB - BA. \tag{2.3}
\]

Let \( K_0 \) be the subfield extension of \( F \) generated over \( K \) by the coefficients of the matrix \( B \). Without loss of generality, we can assume that \( K_0/K \) is a finite Galois extension in the classical sense. We denote by \( \text{Gal}(K_0/K) \) its Galois group and by \( r \) its degree. By [44, Exercise 1.24], there exist unique derivations, still denoted \( \partial \) and \( \delta \) extending \( \partial \) and \( \delta \) to \( K_0 \). Moreover, any element of \( \text{Gal}(K_0/K) \) commutes with the action of \( \delta \) and \( \partial \) on \( K_0 \). Now, let

\[
C := \frac{1}{r} \sum_{\tau \in \text{Gal}(K_0/K)} \tau(B).
\]

Then, \( C \) has coefficients in \( K \) and satisfies

\[
\partial(A) - \delta(C) = \partial(A) - \frac{1}{r} \left( \sum_{\tau \in \text{Gal}(K_0/K)} \tau(\delta(B)) \right) = \partial(A) - \frac{1}{r} \left( \sum_{\tau \in \text{Gal}(K_0/K)} \tau(\delta(A) - BA + AB) \right) = \partial(A) - \delta(A) + CA - AC. \tag{2.4}
\]
This shows that \( \partial(Y) = AY \) is isomonodromic over \( K \). By Proposition 2.46, we find that \( G \) is conjugate to constants in \( \text{GL}_n \).

\[ \square \]

3 Calculating the parameterized Galois group of \( L_1(L_2(y)) = 0 \)

In this section, given two completely reducible \( \delta \)-modules \( L_1 \) and \( L_2 \), we study the parameterized Galois group of a \( \delta \)-module \( \mathcal{U} \), extension of \( L_1 \) by \( L_2 \). In §3.1, we describe \( \text{Gal}^\delta(\mathcal{U}) \) as a semi-direct product of a \( \delta \)-closed subgroup of \( \text{Hom}(\omega(L_1), \omega(L_2)) \) by the parameterized Galois group \( \text{Gal}^\delta(L_1 \oplus L_2) \) (see Theorem 3.3). In §3.2, we perform a first reduction that allows us to set \( L_1 \) equal to the trivial \( \delta \)-module \( I \).

In Theorem 3.13, we show how to deduce the general case from this particular case. In §3.3, we thus focus on the computation of the parameterized Galois group of a \( \delta \)-module \( \mathcal{U} \), extension of \( 1 \) by a completely reducible \( \delta \)-module \( \mathcal{L} \). We then show that one can decompose \( \mathcal{L} \) in a “constant” and a “purely non-constant” part. This decomposition yields a decomposition of \( R_\delta(\text{Gal}^\delta(\mathcal{U})) \). For \( K = \mathbf{k}(x) \), the computation of \( \text{Gal}^\delta(\mathcal{U}) \) in the “constant case” can be deduced from the algorithms contained in [31], whereas the computation of the “purely non-constant” part results from §3.3.2 and the content of Theorem 3.19. Finally, in §3.3.3, we show, under an assumption on \( \mathcal{L} \), that \( R_\delta(\text{Gal}^\delta(\mathcal{U})) \) is the product of the “constant” and “purely non-constant” parts (see Theorem 3.25).

Throughout this section, \( K \) is a \( (\delta, \partial) \)-field of characteristic zero, whose field of \( \delta \)-constants \( \mathbf{k} \) is assumed to be \( \delta \)-closed. We denote also by \( C \) the field of \( \delta \)-constants of \( \mathbf{k} \). We fix a \( \delta \)-fiber functor \( \omega : \text{Diff}_K \to \text{Vec}_C \) on \( \text{Diff}_K \) (see Definition 2.34). Any parameterized Galois group in this section shall be computed with respect to \( \omega \) and is a linear differential algebraic group defined over \( \mathbf{k} \). Any representation is, unless explicitly mentioned, defined over \( \mathbf{k} \).

3.1 Structure of the parameterized Galois group

Let \( L_1, L_2 \in K[\partial] \) be two completely reducible \( \partial \)-operators, and let us denote by \( L_1 \) (respectively, \( L_2 \)) the \( \delta \)-module corresponding to \( L_1(y) = 0 \) (respectively, \( L_2(y) = 0 \)). The \( \delta \)-module \( \mathcal{U} \) over \( K \), corresponding to \( L_1(L_2(y)) = 0 \), is an extension of \( L_1 \) by \( L_2 \),

\[
\begin{array}{ccc}
0 & \longrightarrow & L_2 \\
& i \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \}
Theorem 3.3. Let $\mathcal{L}_1, \mathcal{L}_2$ be two completely reducible $\partial$-modules over $K$. Let $\mathcal{U}$ be a $\partial$-module over $K$, an extension of $\mathcal{L}_1$ by $\mathcal{L}_2$. Then $\text{Gal}^\delta(\mathcal{U})$ is an extension of $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ by a $\partial$-subgroup $W \subset \text{Hom}(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2))$. Moreover, $W$ is stable under the action of $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ on $\text{Hom}(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2))$ given by $g \cdot \phi := g \phi (g^{-1})$ for any $(g, \phi) \in \text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2) \times \text{Hom}(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2))$.

Remark 3.4. The parameterized Galois group $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ acts on the $\delta$-tensor category generated by $\omega(\mathcal{L}_1 \oplus \mathcal{L}_2)$. The $k$-vector space $\text{Hom}(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2))$ belongs to this category, and the action of $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ on $\text{Hom}(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2))$ detailed above is just the description of the Tannakian representation.

Before proving this theorem, we need some intermediate lemmas.

Lemma 3.5. The linear differential algebraic group $\text{Gal}^\delta(\mathcal{U})$ is an extension of the reductive linear differential algebraic group $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ by the linear differential algebraic group $\text{Stab}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$.

Proof. Since $[\mathcal{L}_1 \oplus \mathcal{L}_2]^{*,\delta}$ is a full $\delta$-tensor subcategory of $[\mathcal{U}]^{*,\delta}$, the linear differential algebraic group $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ is a quotient of $\text{Gal}^\delta(\mathcal{U})$. We denote the quotient map by $\pi : \text{Gal}^\delta(\mathcal{U}) \to \text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$. Then $\ker \pi = \text{Stab}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are completely reducible, $\mathcal{L}_1 \oplus \mathcal{L}_2$ is completely reducible as well. This means that the Galois group of $\mathcal{L}_1 \oplus \mathcal{L}_2$ is reductive. Since the latter group is the Zariski closure of $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ inside $\text{GL}(\omega(\mathcal{L}_1 \oplus \mathcal{L}_2))$, [32, Remark 2.9] implies that $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ is a reductive linear differential algebraic group.

Now, we would like to relate $\text{Stab}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ with $R_u(\text{Gal}^\delta(\mathcal{U}))$ and also describe more precisely the structure of the latter groups. By the exactness of $\omega$, $\omega(\mathcal{U})$ is an extension of $\omega(\mathcal{L}_1)$ by $\omega(\mathcal{L}_2)$ in the category of representations of $\text{Gal}^\delta(\mathcal{U})$.

Lemma 3.6. In the above notation, let $s$ be a $k$-linear section of the exact sequence:

$$0 \longrightarrow \omega(\mathcal{L}_2) \xrightarrow{\omega(i)} \omega(\mathcal{U}) \xrightarrow{\omega(p)} \omega(\mathcal{L}_1) \longrightarrow 0. \tag{3.1}$$

We consider the following map

$$\zeta_\mathcal{U} : \text{Gal}^\delta(\mathcal{U}) \to \text{Hom}(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2)), \quad g \mapsto \left\{ x \mapsto g(s(g^{-1}x)) - s(x) \right\}.$$

Then the restriction of the map $\zeta_\mathcal{U}$ onto $\text{Stab}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ is an injective morphism of linear differential algebraic groups. Moreover, the linear differential algebraic group $\text{Stab}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ is abelian and coincides with the unipotent radical $R_u(\text{Gal}^\delta(\mathcal{U}))$ of $\text{Gal}^\delta(\mathcal{U})$.

Proof. For all $g_1, g_2 \in \text{Gal}^\delta(\mathcal{U})$, we have:

$$\zeta_\mathcal{U}(g_1 g_2)(x) = g_1 \zeta_\mathcal{U}(g_2)(g_1^{-1}x) + \zeta_\mathcal{U}(g_1)(x). \tag{3.2}$$

If $g_1, g_2 \in \text{Stab}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$, equation (3.2) gives

$$\zeta_\mathcal{U}(g_1 g_2) = \zeta_\mathcal{U}(g_1) + \zeta_\mathcal{U}(g_2).$$

This means that $\zeta_\mathcal{U}$ is a morphism of linear differential algebraic groups from $\text{Stab}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ to $\text{Hom}(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2))$.

Moreover, let $\{e_j\}_{j=1}^s$ (resp., $\{f_i\}_{i=1}^r$) be a $k$-basis of $\omega(\mathcal{L}_2)$ (resp. $\omega(\mathcal{L}_1)$). Then

$$\{\omega(i)(e_j), s(f_j)\}_{i=1,...,s, j=1,...,r}$$
is a \(k\)-basis of \(\omega(\mathcal{U})\). If \(g \in \text{Stab}^0(\mathcal{L}_1 \oplus \mathcal{L}_2) \cap \ker(\zeta_\mathcal{U})\), then \(g\) induces the identity on

\[
\{\omega(i)(e_i), s(f_j)\}_{i=1,...,r, j=1,...,r}
\]

and thereby on \(\omega(\mathcal{U})\). Hence, by definition of \(\text{Gal}^0(\mathcal{U})\), the element \(g\) is the identity and, therefore, \(\ker \left( \zeta_\mathcal{U} \big|_{\text{Stab}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)} \right)\) is trivial.

Since \(\text{Hom}(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2))\) is abelian, the same holds for \(\text{Stab}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)\). Moreover, \(\text{Stab}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)\) is unipotent. Indeed, let \(e\) be the neutral element in \(\text{Gal}^0(\mathcal{U})\). Let \(x\) be in \(\omega(\mathcal{L}_1)\) and \(g \in \text{Stab}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)\). Since \(g \cdot s(x) - s(x) \in \omega(\mathcal{L}_2)\), we have

\[
(g - e)^2(s(x)) = (g - e)(g \cdot s(x) - s(x)) = g(g \cdot s(x) - s(x)) - (g \cdot s(x) - s(x)) = 0.
\]

Reasoning as above, we find that \((g - e)^2\) is zero on \(\omega(\mathcal{U})\). By Lemma 3.2, \(\text{Stab}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)\) is also normal and, hence, must be contained in \(R_u(\text{Gal}^0(\mathcal{U}))\). By [8, Theorem 1], the image of a unipotent linear differential algebraic group is unipotent. By Lemma 3.5, \(\text{Stab}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)\) is the kernel of the projection of \(\text{Gal}^0(\mathcal{U})\) on the reductive linear differential algebraic group \(\text{Gal}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)\). It follows that \(R_u(\text{Gal}^0(\mathcal{U}))\) is contained in \(\text{Stab}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)\), which ends the proof.

**Remark 3.7.** Since two sections of (3.1) differ by a map from \(\omega(\mathcal{L}_1)\) to \(\omega(\mathcal{L}_2)\), one sees that, when restricted to \(R_u(\text{Gal}^0(\mathcal{U}))\), the map \(\zeta_\mathcal{U}\) is independent of the choice of the section.

By the above lemma, \(R_u(\text{Gal}^0(\mathcal{U}))\) is an abelian normal subgroup of \(\text{Gal}^0(\mathcal{U})\). Since \(\text{Gal}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)\) is the quotient of \(\text{Gal}^0(\mathcal{U})\) by \(R_u(\text{Gal}^0(\mathcal{U}))\) and \(R_u(\text{Gal}^0(\mathcal{U}))\) is abelian, the linear differential algebraic group \(\text{Gal}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)\) acts by conjugation on \(R_u(\text{Gal}^0(\mathcal{U}))\). The lemma below shows that this action is compatible with the action of \(\text{Gal}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)\) on \(\text{Hom}_k(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2))\).

**Lemma 3.8.** For all \(g_1 \in \text{Gal}^0(\mathcal{U})\), \(g_2 \in R_u(\text{Gal}^0(\mathcal{U}))\), and \(x \in \omega(\mathcal{L}_1)\), we have

\[
\zeta_\mathcal{U}(g_1 g_2 g_1^{-1})(x) = g_1 \zeta_\mathcal{U}(g_2)(g_1^{-1} x) = g_1 \zeta_\mathcal{U}(g_2) x,
\]

where the \(*\) denotes the natural action of \(\text{Gal}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)\) on \(\text{Hom}(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2))\) via \(g \ast \phi := g \circ \phi \circ g^{-1}\) for \(\phi \in \text{Hom}(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2))\) and \(g \in \text{Gal}^0(\mathcal{L}_1 \oplus \mathcal{L}_2)\).

**Proof.** Let \(e\) denote the neutral element in \(\text{Gal}^0(\mathcal{U})\). From (3.2), we find that, for all \(x \in \omega(\mathcal{L}_1)\),

\[
g_1 \zeta_\mathcal{U}(g_1^{-1})(g_1^{-1} x) = \zeta_\mathcal{U}(e)(x) - \zeta_\mathcal{U}(g_1)(x) = -\zeta_\mathcal{U}(g_1)(x).
\]

Applying repeatedly (3.2), we find that for all \(x \in \omega(\mathcal{L}_1)\),

\[
\zeta_\mathcal{U}(g_1 g_2 g_1^{-1})(x) = g_1 \zeta_\mathcal{U}(g_2 g_1^{-1})(g_1^{-1} x) + \zeta_\mathcal{U}(g_1)(x)
\]

\[
= g_1 (g_2 \zeta_\mathcal{U}(g_1^{-1})(g_1^{-1} g_2 g_1^{-1} x) + \zeta_\mathcal{U}(g_2)(g_1^{-1} x)) + \zeta_\mathcal{U}(g_1)(x)
\]

\[
= g_1 \zeta_\mathcal{U}(g_2)(g_1^{-1} x) + g_1 g_2 g_1^{-1} (g_1 \zeta_\mathcal{U}(g_1^{-1})(g_1^{-1} g_2 g_1^{-1} x)) + \zeta_\mathcal{U}(g_1)(x),
\]

for all \(x \in \omega(\mathcal{L}_1)\). Since

\[
g_1 g_2 g_1^{-1}, g_1 g_2 g_1^{-1} \in R_u(\text{Gal}^0(\mathcal{U})) = \text{Stab}^0(\mathcal{L}_1 \oplus \mathcal{L}_2),
\]

we get that, for all \(x \in \omega(\mathcal{L}_1)\),

\[
g_1 g_2 g_1^{-1} (g_1 \zeta_\mathcal{U}(g_1^{-1})(g_1^{-1} g_2 g_1^{-1} x)) + \zeta_\mathcal{U}(g_1)(x) = g_1 \zeta_\mathcal{U}(g_1^{-1})(g_1^{-1} x) + \zeta_\mathcal{U}(g_1)(x) = 0.
\]

We conclude that, for all \(x \in \omega(\mathcal{L}_1)\), \(\zeta_\mathcal{U}(g_1 g_2 g_1^{-1})(x) = g_1 \zeta_\mathcal{U}(g_2)(g_1^{-1} x)\).
Proof of Theorem 3.3. By the above, $\text{Gal}^\delta(\mathcal{U})$ is an extension of $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ by $R_u(\text{Gal}^\delta(\mathcal{U}))$. The action of $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ on $R_u(\text{Gal}^\delta(\mathcal{U}))$ is deduced from the action by conjugation of $\text{Gal}^\delta(\mathcal{U})$ on its unipotent radical.

Combining Lemma 3.6 and Lemma 3.8, we can identify via $\zeta_u$, the unipotent radical $R_u(\text{Gal}^\delta(\mathcal{U}))$ with a $\delta$-closed subgroup of $\text{Hom}(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2))$ and the action of $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ on $R_u(\text{Gal}^\delta(\mathcal{U}))$ by conjugation with the action of $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ on $\text{Hom}(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2))$, induced by the representation of $\text{Gal}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ on $\omega(\mathcal{L}_1 \oplus \mathcal{L}_2)$.

Remark 3.9. The extension in Theorem 3.3 does not split in general. For example,

$$G = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_3(k) \middle| \delta(b) = \frac{\delta(a)}{a} \right\}$$

is a linear differential algebraic group such that the quotient map $G \to G/R_u(G) \cong k^\times$ does not have a $\delta$-polynomial section. Indeed, otherwise, $G$ would have a projection onto $R_u(G) \cong C$, which is impossible because $G$ is strongly connected [10, Example 2.25].

Remark 3.10. If $K = k(x)$ and $\delta := \frac{\partial}{\partial x}$, the knowledge of $R := R_u(\text{Gal}^\delta(\mathcal{U}))$ allows one to compute $G := \text{Gal}^\delta(\mathcal{U})$ algorithmically. Indeed, first of all, one can compute the normalizer $N$ of $R$ in $\text{GL}(\omega(\mathcal{U}))$. Note that $G \subset N$. Second, by the differential version of Chevalley theorem [29, Theorem 5.1] (see also [5, proof of Theorem 5.6]), there is $\mathcal{U}_0 \in \{\mathcal{U}\}^{\delta, \delta}$ and a differential representation $\rho : N \to \text{GL}(\omega(\mathcal{U}_0))$ such that $R = \ker \rho$. The proof of this Chevalley theorem leads to a constructive procedure to find $\mathcal{U}_0$ and $\rho$. Since $\text{Gal}^\delta(\mathcal{U}_0) = \rho(G)$ is reductive, one can compute it [32].

Now, we can find $G$ as $\rho^{-1}(\text{Gal}^\delta(\mathcal{U}_0))$.

In view of Remark 3.10, we want to compute the parameterized Galois group of $\mathcal{U}$. For this, we will perform a first reduction that will allow us to simplify our computation.

3.2 A first reduction

Let $L_1, L_2 \in K[\partial]$ be two completely reducible $\partial$-operators. Let us denote the $\partial$-module over $K$ corresponding to $L_1(y) = 0$ (resp., $L_2(y) = 0$) by $\mathcal{L}_1$ (resp., by $\mathcal{L}_2$). The $\partial$-module $\mathcal{U}$ corresponding to $L_1(L_2(y)) = 0$ is an extension of $\mathcal{L}_1$ by $\mathcal{L}_2$,

$$0 \longrightarrow \mathcal{L}_2 \overset{i}{\longrightarrow} \mathcal{U} \overset{p}{\longrightarrow} \mathcal{L}_1 \longrightarrow 0,$$

in the category of $\partial$-modules over $K$. In this section, we recall the methods of [4] to show that one can reduce our study to the case in which $L_1$ is of the form $\partial - \frac{\partial b}{\partial}$ for some $b \in K^\times$.

We first describe the reduction process in terms of $\partial$-modules. Since the functor $\underline{\text{Hom}}(\mathcal{L}_1, -)$ is exact, we can construct from (3.4) the exact sequence:

$$0 \longrightarrow \text{Hom}(\mathcal{L}_1, \mathcal{L}_2) \longrightarrow \text{Hom}(\mathcal{L}_1, \mathcal{U}) \longrightarrow \text{Hom}(\mathcal{L}_1, \mathcal{L}_1) \longrightarrow 0,$$

(3.5)

We pull back (3.5) by the diagonal embedding

$$d : 1 \to \text{Hom}(\mathcal{L}_1, \mathcal{L}_1), \quad \lambda \mapsto \lambda \text{id}_\mathcal{L}_1,$$

where $1$ is the trivial $\partial$-module. We obtain an exact sequence

$$0 \longrightarrow \text{Hom}(\mathcal{L}_1, \mathcal{L}_2) \longrightarrow \mathcal{R}(\mathcal{U}) \longrightarrow 1 \longrightarrow 0,$$

(3.6)

where $\mathcal{R}(\mathcal{U})$ is the $\partial$-module deduced from $\mathcal{U}$ by the pullback. We call the $\partial$-module $\mathcal{R}(\mathcal{U})$ the reduction of $\mathcal{U}$. We recall that, as a $K$-vector space, $\mathcal{R}(\mathcal{U})$ coincides with the set

$$\left\{ (\phi, \lambda) \in \text{Hom}(\mathcal{L}_1, \mathcal{U}) \times 1 \middle| p \circ \phi = \lambda \text{id}_\mathcal{L}_1 \right\}.$$
**Remark 3.11.** An effective interpretation of this reduction process in terms of matrix differential equations immediately follows from [4, page 15].

**Proposition 3.12.** In the above notation, we have

1. The parameterized Galois group \( \text{Gal}^\delta(\text{Hom}(L_1, L_2)) \) is a quotient of \( \text{Gal}^\delta(L_1 \oplus L_2) \) and is a reductive linear differential algebraic group;

2. By Lemma 3.6, one can identify \( R_u(\text{Gal}^\delta(\mathcal{U})) \) (respectively, \( R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) \)) with a differential algebraic subgroup of \( \text{Hom}(\omega(L_1), \omega(L_2)) \) (respectively, of \( \text{Hom}(k, \text{Hom}(\omega(L_1), \omega(L_2))) \)). Then the canonical isomorphism

   \[
   \phi : \text{Hom}(k, \text{Hom}(\omega(L_1), \omega(L_2))) \to \text{Hom}(\omega(L_1), \omega(L_2)), \quad \psi \to \psi(1)
   \]

   induces an isomorphism of linear differential algebraic groups between the unipotent radical of \( \text{Gal}^\delta(\mathcal{U}) \) and the unipotent radical of \( \text{Gal}^\delta(\mathcal{R}(\mathcal{U})) \);

3. By Lemma 3.8, \( \text{Gal}^\delta(L_1 \oplus L_2) \) (respectively, \( \text{Gal}^\delta(\text{Hom}(L_1, L_2)) \)) acts on \( R_u(\text{Gal}^\delta(\mathcal{U})) \) (respectively, on \( R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) \)). These actions are compatible with the isomorphism \( \phi \).

**Proof.**

1. Since \( \text{Hom}(L_1, L_2) \) (respectively, \( L_1 \oplus L_2 \)) is a sub-object of \( \mathcal{U}^{*, \delta} \), its parameterized Galois group is a quotient of \( \text{Gal}^\delta(\mathcal{U}) \) by \( \text{Stab}^\delta(\text{Hom}(L_1, L_2)) \) (respectively, by \( \text{Stab}^\delta(L_1 \oplus L_2) = \text{Stab}^\delta(L_1) \cap \text{Stab}^\delta(L_2) \)). Now, the inclusion

   \[
   \text{Stab}^\delta(L_1 \oplus L_2) \subset \text{Stab}^\delta(\text{Hom}(L_1, L_2))
   \]

   is obvious. Since stabilizers of objects in \( \mathcal{U}^{*, \delta} \) are normal in \( \text{Gal}^\delta(\mathcal{U}) \) by Lemma 3.2, we can apply [8, Proposition 2] to get that

   \[
   \text{Gal}^\delta(\text{Hom}(L_1, L_2)) = \frac{\text{Gal}^\delta(\mathcal{U})}{\text{Stab}^\delta(\text{Hom}(L_1, L_2))}
   \]

   is a quotient of

   \[
   \text{Gal}^\delta(L_1 \oplus L_2) = \frac{\text{Gal}^\delta(\mathcal{U})}{\text{Stab}^\delta(L_1 \oplus L_2)}
   \]

   by

   \[
   \frac{\text{Stab}^\delta(\text{Hom}(L_1, L_2))}{\text{Stab}^\delta(L_1 \oplus L_2)}.
   \]

   The same reasoning in the non-parameterized case shows that \( \text{Gal}(\text{Hom}(L_1, L_2)) \) is a quotient of \( \text{Gal}(L_1 \oplus L_2) \). Since quotients of reductive algebraic groups are reductive, [32, Remark 2.9] allows us to conclude that \( \text{Gal}^\delta(\text{Hom}(L_1, L_2)) \) is a reductive linear differential algebraic group.

2. Since \( \mathcal{R}(\mathcal{U}) \) is an object of \( \mathcal{U}^{*, \delta} \), \( \text{Gal}^\delta(\mathcal{R}(\mathcal{U})) \) is a quotient of \( \text{Gal}^\delta(\mathcal{U}) \), and we denote the canonical surjection by \( \pi \). The image of \( \text{Stab}^\delta(\text{Hom}(L_1, L_2)) \) via \( \pi \) coincides with the stabilizer of \( \text{Hom}(L_1, L_2) \) inside \( \text{Gal}^\delta(\mathcal{R}(\mathcal{U})) \) and, thus, with \( R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) \) by Lemmas 3.5 and 3.6.

   Let \( H \subset R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) \) be the image of \( \text{Stab}^\delta(L_1 \oplus L_2) \) by \( \pi \). By [6, Proposition 7, page 908], \( H \) is a differential algebraic subgroup of \( R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) \). Since \( \text{Stab}^\delta(L_1 \oplus L_2) \) is normal in \( \text{Gal}^\delta(\mathcal{U}) \) and \( \pi \) is surjective, \( H \) is normal in \( R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) \), and we can consider the quotient map

   \[
   p : R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) \to R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) / H.
   \]

   Since quotients of unipotent linear differential algebraic groups are unipotent by [8, Theorem 1], the linear differential algebraic group \( R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) / H \) is unipotent. Now,

   \[
   R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) / H = \pi(\text{Stab}^\delta(\text{Hom}(L_1, L_2))) / \pi(\text{Stab}^\delta(L_1 \oplus L_2)).
   \]

   (3.7)
The surjective morphism $\pi$ is induced via $\delta$-Tannakian equivalence by the inclusion of $\delta$-Tannakian categories $(\mathcal{A}(\mathcal{U}))^{\delta,\delta} \subset (\mathcal{U})^{\delta,\delta}$. This inclusion restricts to the inclusion of usual Tannakian categories $(\mathcal{R}(\mathcal{U}))^{\delta} \subset (\mathcal{U})^{\delta}$. This shows that $\pi$ extends in a compatible way to the Zariski closure to the surjective morphism of algebraic groups $\overline{\pi}: \text{Gal}(\mathcal{U}) \rightarrow \text{Gal}(\mathcal{R}(\mathcal{U}))$. One can show that

\[
\overline{\pi}(\text{Stab}(\text{Hom}(\mathcal{L}_1, \mathcal{L}_2)))/\overline{\pi}(\text{Stab}(\mathcal{L}_1 \oplus \mathcal{L}_2))
\]

coincides with the Zariski closure of $R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) / H$.

Let $K_{\mathcal{L}_1 \oplus \mathcal{L}_2}$ (respectively, $K_{\text{Hom}(\mathcal{L}_1, \mathcal{L}_2)}$) denote the usual PV field of $\mathcal{L}_1 \oplus \mathcal{L}_2$ (respectively, of $\text{Hom}(\mathcal{L}_1, \mathcal{L}_2)$). Let $K_{\mathcal{U}}$ (respectively, $K_{\mathcal{R}(\mathcal{U})}$) denote the usual PV field of $\mathcal{U}$ (respectively, of $\mathcal{R}(\mathcal{U})$). We have the following tower of $\delta$-field extensions:

\[
\begin{array}{ccc}
K_{\mathcal{U}} & \rightarrow & K_{\mathcal{R}(\mathcal{U})} \\
\downarrow & & \downarrow \\
K_{\text{Hom}(\mathcal{L}_1, \mathcal{L}_2)} & \rightarrow & K_{\mathcal{L}_1 \oplus \mathcal{L}_2} \\
\downarrow & & \downarrow \\
K & & K
\end{array}
\]

Now, we see that

\[
\text{Gal}\left(K_{\mathcal{L}_1 \oplus \mathcal{L}_2} / K_{\text{Hom}(\mathcal{L}_1, \mathcal{L}_2)}\right) = \text{Stab}(\text{Hom}(\mathcal{L}_1, \mathcal{L}_2)) / \text{Stab}(\mathcal{L}_1 \oplus \mathcal{L}_2).
\]

Since $K_{\text{Hom}(\mathcal{L}_1, \mathcal{L}_2)} / K$ is a PV-extension, the group $\text{Gal}\left(K_{\mathcal{L}_1 \oplus \mathcal{L}_2} / K_{\text{Hom}(\mathcal{L}_1, \mathcal{L}_2)}\right)$ is normal in $\text{Gal}(K_{\mathcal{L}_1 \oplus \mathcal{L}_2} / K)$ by the PV-correspondence. Therefore, $\text{Gal}\left(K_{\mathcal{L}_1 \oplus \mathcal{L}_2} / K_{\text{Hom}(\mathcal{L}_1, \mathcal{L}_2)}\right)$ is a reductive algebraic group. Since

\[
\overline{\pi}: \text{Stab}(\text{Hom}(\mathcal{L}_1, \mathcal{L}_2)) / \text{Stab}(\mathcal{L}_1 \oplus \mathcal{L}_2) \rightarrow \overline{\pi}(\text{Stab}(\text{Hom}(\mathcal{L}_1, \mathcal{L}_2)))/\overline{\pi}(\text{Stab}(\mathcal{L}_1 \oplus \mathcal{L}_2))
\]

is a quotient map, we find from the above identifications that the Zariski closure of $R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) / H$ is a reductive algebraic group. We conclude by [32, Remark 2.9] that $R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) / H$ is reductive. On the other hand, since $R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))) / H$ is both unipotent and reductive, it must be trivial, and we find that

\[
\pi(\text{Stab}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)) = \pi(\text{Stab}(\text{Hom}(\mathcal{L}_1, \mathcal{L}_2))) = R_u(\text{Gal}^\delta(\mathcal{R}(\mathcal{U}))).
\] (3.8)

We recall the notation of Lemma 3.6. We denote by $s$ a $k$-linear section of the exact sequence of finite-dimensional representations of $\text{Gal}^\delta(\mathcal{U})$:

\[
0 \rightarrow \omega(\mathcal{L}_2) \xrightarrow{\omega(i)} \omega(\mathcal{U}) \xrightarrow{\omega(p)} \omega(\mathcal{L}_1) \rightarrow 0.
\]

Then, we identify $R_u(\text{Gal}^\delta(\mathcal{U})) = \text{Stab}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ with the image of $\text{Stab}^\delta(\mathcal{L}_1 \oplus \mathcal{L}_2)$ via

\[
\xi:\ R_u(\text{Gal}^\delta(\mathcal{U})) \rightarrow \text{Hom}_k(\omega(\mathcal{L}_1), \omega(\mathcal{L}_2)), \quad g \mapsto \{x \mapsto gs(g^{-1}x) - s(x)\}.
\]
Now, since $\omega$ is compatible with $\mathrm{Hom}$, the map $r : k \rightarrow \omega(\mathcal{R}(\mathcal{U})), \lambda \mapsto (\lambda s, \lambda)$, is a $k$-linear section of

$$0 \longrightarrow \mathrm{Hom}(\omega(L_1), \omega(L_2)) \longrightarrow \omega(\mathcal{R}(\mathcal{U})) \xrightarrow{r \cdot \ldots \cdot} k \longrightarrow 0.$$  

We apply again Lemma 3.6 to identify $R_u(\mathrm{Gal}^\delta(\mathcal{R}(\mathcal{U}))) = \pi\{\mathrm{Stab}^\delta(L_1 \oplus L_2)\}$ with its image via

$$\zeta_{\mathcal{R}(\mathcal{U})} : \mathrm{Gal}^\delta(\mathcal{R}(\mathcal{U})) \rightarrow \mathrm{Hom}(k, \mathrm{Hom}_k(\omega(L_1), \omega(L_2))), \quad g \mapsto (\lambda \mapsto g r(\lambda) g^{-1} - r(\lambda)).$$

Identifying $\mathrm{Hom}(k, \mathrm{Hom}(\omega(L_1), \omega(L_2)))$ with $\mathrm{Hom}(\omega(L_1), \omega(L_2))$ via $\phi$, we find that

$$\zeta_\mathcal{U} = \phi \circ \zeta_{\mathcal{R}(\mathcal{U})} \circ \pi.$$  \hspace{1cm} (3.9)

Now, we have

$$R_u(\mathrm{Gal}^\delta(\mathcal{U})) = \zeta_\mathcal{U}(\mathrm{Stab}^\delta(L_1 \oplus L_2)) = \zeta_{\mathcal{R}(\mathcal{U})} \circ \pi(\mathrm{Stab}^\delta(L_1 \oplus L_2)) = R_u(\mathrm{Gal}^\delta(\mathcal{R}(\mathcal{U}))),$$

where we have used Remark 3.7.

(3) The compatibility of the actions comes from Lemma 3.8, (3.9), and (3.8). \hfill $\blacksquare$

We combine Proposition 3.12 and Theorem 3.3 in the following Theorem.

**Theorem 3.13.** Let $L_1, L_2$ be two completely reducible $\delta$-modules over $K$ and let $\mathcal{U}$ be a $\delta$-module extension of $L_1$ by $L_2$. Then, $\mathrm{Gal}^\delta(\mathcal{U})$ is an extension of $\mathrm{Gal}^\delta(L_1 \oplus L_2)$ by a $\delta$-subgroup $W$ of $\omega(\mathrm{Hom}(L_1, L_2))$. Moreover, $W = R_u(\mathrm{Gal}^\delta(\mathcal{R}(\mathcal{U})))$, where $\mathcal{R}(\mathcal{U})$ is an extension of $1$ by the completely reducible $\delta$-module $\mathrm{Hom}(L_1, L_2)$, and the action of $\mathrm{Gal}^\delta(L_1 \oplus L_2)$ on $W$ is given by composing the quotient map of $\mathrm{Gal}^\delta(L_1 \oplus L_2)$ on $\mathrm{Gal}^\delta(\mathrm{Hom}(L_1, L_2))$ with the action of $\mathrm{Gal}^\delta(\mathrm{Hom}(L_1, L_2))$ on $\omega(\mathrm{Hom}(L_1, L_2))$.

### 3.3 Unipotent radical of parameterized Galois group of extension of $1$ by $\mathcal{L}$

Let $\mathcal{L}$ be a completely reducible $\delta$-module over $K$ and $\mathcal{U}$ be an extension of $1$ by $\mathcal{L}$. In this section, we study $R_u(\mathrm{Gal}^\delta(\mathcal{U}))$.

In terms of $\delta$-operators, the situation corresponds to the following. Let $L \in K[\delta]$ be a completely reducible $\delta$-operator and $\mathcal{L}$ be the associated $\delta$-module. An extension $\mathcal{U}$ of $1$ by $\mathcal{L}$ corresponds to an inhomogeneous differential equation of the form $L(y) = b$ for some $b \in K^\ast$. The main result of [4] is to show that $R_u(\mathrm{Gal}(\mathcal{U})) = \omega(\mathcal{L}_0)$, where $\mathcal{L}_0$ is the largest $\delta$-module of $\mathcal{L}$ such that

1. $L = L_1 L_0$;
2. $L_1(y) = b$ has a solution in $K$.

From Lemma 3.6, we know that $R_u(\mathrm{Gal}^\delta(\mathcal{U}))$ can be identified with a differential algebraic subgroup $W$ of $\omega(\mathcal{L}_0)$, stable under the natural action of $\mathrm{Gal}^\delta(\mathcal{L})$ on $\omega(\mathcal{L})$. In [19], the result of [4] was rephrased in Tannakian terms and it was proved that $\mathcal{L}_0$ is the smallest sub-object of $\mathcal{L}$ such that the pushout of the extension of $\mathcal{U}$ by the quotient map $\pi : \mathcal{L} \rightarrow \mathcal{L} \backslash \mathcal{L}_0$ is a trivial extension. Such a characterization no longer holds in general in the parameterized setting. Indeed, the classification of differential algebraic subgroups of vector groups shows that $W$ coincides with the zero set of a finite collection of linear homogeneous differential equations with coefficients in $k$. Then, we have two options:
In the first case, we deduce from the $\delta$-Tannakian equivalence that $W = \omega(\mathcal{L}_0)$ for a submodule $\mathcal{L}_0$ of $\mathcal{L}$ if and only if it is an algebraic subgroup of $\omega(\mathcal{L})$. In this situation, we show that $\mathcal{L}_0$ is the smallest $\delta$-submodule of $\mathcal{L}$ such that the parameterized Galois group of the pushout of the extension $\mathcal{U}$ by the quotient map $\pi : \mathcal{L} \to \mathcal{L}/\mathcal{L}_0$ has trivial unipotent radical (see Theorem 3.19). This last condition can be tested due to an algorithm contained in [32].

If $W$ is not given by linear homogeneous $\delta$-polynomials of order 0, then $W$ is not of the form $\omega(\mathcal{L})$ for any $\mathcal{L}$. Also, the order of the defining equations of $W$ can be as high as desired even for second order differential equations:

**Example 3.14.** For $n \geq 0$, let

$$z(x, t, n) := \sum_{j=0}^{n} t^j \ln(x + j); \quad a(x, t, n) := \frac{\partial z(x, t, n)}{\partial x} = \sum_{j=0}^{n} \frac{t^j}{x + j} \in \mathbf{k}(x),$$

where $\mathbf{k}$ is a differentially closed field with respect to $\partial/\partial t$ containing $\mathbf{Q}(t)$. Then the function $z(x, t, n)$ satisfies the following second order differential equation in $y(x, t)$ over $\mathbf{k}(x)$:

$$\frac{\partial}{\partial x} \left( \frac{\partial y(x, t)}{\partial x} / a(x, t, n) \right) = 0 \iff \frac{\partial^2 y(x, t)}{\partial x^2} - \frac{\partial a(x, t, n)}{\partial x} \frac{\partial y(x, t)}{\partial x} = 0.$$

Since $\ln(x), \ldots, \ln(x + n)$ are algebraically independent over $\mathbf{k}(x)$ by [38, 14], $\frac{\partial^{n+1} z(x, t, n)}{\partial t^{n+1}} = 0$, and

$$\mathbf{k}(x)(\ln(x), \ldots, \ln(x + n)) = \mathbf{k}(x) \left( \frac{\partial^j (z(x, t, n))}{\partial t^j} \bigg| j \geq 0 \right),$$

we have

$$\text{Gal}^\delta = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \bigg| \frac{\partial^{n+1} a}{\partial t^{n+1}} = 0 \right\}.$$
\[ \text{Let } \mathcal{L}_c, \mathcal{L}_n \text{ be irreducible } \partial\text{-submodules such that } \mathcal{L} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r. \text{ We have} \]

\[ \text{GL}(\omega(\mathcal{L})) = \prod_{i=1}^{r} \text{GL}(\omega(\mathcal{L}_i)). \]

Let \( S \) be the set of indices \( i \) in \( \{1, \ldots, r\} \) such that the representation of Gal\( ^\delta(\mathcal{L}) \) on \( \omega(\mathcal{L}_i) \) is conjugate to constants inside GL(\( \omega(\mathcal{L}_i) \)). Setting

\[ \mathcal{L}_c := \bigoplus_{i \in S} \mathcal{L}_i \text{ and } \mathcal{L}_{nc} = \bigoplus_{i \notin S} \mathcal{L}_i \]

allows to conclude the proof.

\[ \square \]

**Remark 3.16.** The above construction is effective. Let \( \mathcal{L} \) be a completely reducible \( \partial \)-module over \( K = \mathbb{C}(z) \) with \( \partial(z) = 1 \) and \( \partial(\mathcal{C}) = 0 \). There are many algorithms that compute a factorization of \( \mathcal{L} \) into a direct sum of irreducible \( \partial \)-submodules: see, for instance, [45, 42]. Thus, we can find a linear differential system associated to \( \mathcal{L} \) of the form

\[ \partial(Y) = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & A_r \end{pmatrix} Y \]

with \( A_i \in K^{n_i \times n_i} \) for all \( i = 1, \ldots, r \) and \( \partial(Y) = A_i Y \) is an irreducible differential system. For all \( i = 1, \ldots, r \), let \( \mathcal{L}_i \) be a \( \partial \)-module associated to \( \partial(Y) = A_i Y \). Let \( S \) be the set of indices \( i \) such that there exists a matrix \( B_i \in K^{n_i \times n_i} \) such that

\[ \delta(A_i) - \partial(B_i) = B_i A_i - A_i B_i. \]

Since there are algorithms to find rational solutions of linear differential systems, the construction of the set \( S \) is also effective. Now, we can set \( \mathcal{L}_c := \bigoplus_{i \in S} \mathcal{L}_i \) and \( \mathcal{L}_{nc} = \bigoplus_{i \notin S} \mathcal{L}_i \).

This decomposition motivates the following definition.

**Definition 3.17.** A \( \partial \)-module \( \mathcal{L} \) over \( K \) is called constant if the representation of Gal\( ^\delta(\mathcal{L}) \) on \( \omega(\mathcal{L}) \) is conjugate to constants in GL(\( \omega(\mathcal{L}) \)). On the contrary, the \( \partial \)-module \( \mathcal{L} \) is called purely non-constant if there is no nontrivial \( \partial \)-submodule \( \mathcal{N} \) of \( \mathcal{L} \) such that the representation of Gal\( ^\delta(\mathcal{L}) \) on \( \omega(\mathcal{N}) \) is conjugate to constants inside GL(\( \omega(\mathcal{N}) \)).

**Remark 3.18.** We say that a \( G \)-module \( V \) is purely non-constant if, for every nontrivial \( G \)-submodule \( W \) of \( V \), the induced representation \( p : G \rightarrow \text{GL}(W) \) is non-constant. By the Tannakian equivalence, a \( \partial \)-module \( \mathcal{L} \) is purely non-constant if and only if the Gal\( ^\delta(\mathcal{L}) \)-module \( \omega(\mathcal{L}) \) is purely non-constant.

Recall that \( \mathcal{U} \) is a \( \partial \)-module extension of \( 1 \) by \( \mathcal{L} \). Now, we consider the pushout of

\[ \begin{array}{ccc} 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{U} & \longrightarrow & 1 & \longrightarrow & 0 \end{array} \]

by the projection of \( \mathcal{L} \) on \( \mathcal{L}_c \) (respectively, on \( \mathcal{L}_{nc} \)). We find two exact sequences of \( \partial \)-modules:

\[ \begin{array}{ccc} 0 & \longrightarrow & \mathcal{L}_c & \longrightarrow & \mathcal{U}_c & \longrightarrow & 1 & \longrightarrow & 0 \end{array}, \quad \text{and} \quad \begin{array}{ccc} 0 & \longrightarrow & \mathcal{L}_{nc} & \longrightarrow & \mathcal{U}_{nc} & \longrightarrow & 1 & \longrightarrow & 0 \end{array}. \]

We deduce from Lemma 3.6 that
• \( R_u(\text{Gal}^\delta(\mathcal{U})) \) is a differential algebraic subgroup of \( \omega(\mathcal{L}) \);
• \( R_u(\text{Gal}^\delta(\mathcal{U}_c)) \) is a differential algebraic subgroup of \( \omega(\mathcal{L}_c) \);
• \( R_u(\text{Gal}^\delta(\mathcal{U}_{nc})) \) is a differential algebraic subgroup of \( \omega(\mathcal{L}_{nc}) \).

The quotient \( \text{Gal}^\delta(\mathcal{U}_c)/R_u(\text{Gal}^\delta(\mathcal{U}_c)) \) is \( \text{Gal}^\delta(\mathcal{L}_c) \), which is, by construction, conjugate to constants. We can use [31] to compute \( R_u(\text{Gal}^\delta(\mathcal{U}_c)) \). The section 3.3.2 shows how to compute the unipotent radical of the parameterized Galois group of an extension of \( 1 \) by a purely non constant completely reducible module. Finally §3.3.3 shows how to combine §3.3.2 with [31] to deduce \( R_u(\text{Gal}^\delta(\mathcal{U})) \) from the computation of \( R_u(\text{Gal}^\delta(\mathcal{U}_c)) \) and \( R_u(\text{Gal}^\delta(\mathcal{U}_{nc})) \).

### 3.3.2 The purely non-constant case

The aim of this section is to prove the following theorem.

**Theorem 3.19.** Let \( \mathcal{L} \) be a purely non-constant completely reducible \( \partial \)-module over \( K \). Let \( \mathcal{U} \) be a \( \partial \)-module, an extension of \( 1 \) by \( \mathcal{L} \). Then, \( R_u(\text{Gal}^\delta(\mathcal{U})) = \omega(\mathcal{L}_0) \), where \( \mathcal{L}_0 \) is the smallest \( \partial \)-submodule of \( \mathcal{L} \) such that \( \text{Gal}^\delta(\mathcal{U}/\mathcal{L}_0) \) is reductive.

By Theorem 3.13, \( R_u(\text{Gal}^\delta(\mathcal{U})) \) is a \( \delta \)-closed subgroup \( W \) of \( \omega(\mathcal{L}) \) stable under the action of \( \text{Gal}^\delta(\mathcal{L}) \). We show that any \( \delta \)-subgroup \( W \) of \( \omega(\mathcal{L}) \) stable under the action of \( \text{Gal}^\delta(\mathcal{L}) \) is a \( K \)-vector space. In this attempt, we first treat the cases in which \( \text{Gal}^\delta(\mathcal{L}) \) is a torus or \( \text{SL}_2 \). We conclude with the general situation and the proof of Theorem 3.19.

The algorithm contained in [32] allows to test whether the unipotent radical of a linear algebraic group is zero. This algorithm relies on bounds on the order of the defining equations of the parameterized Galois group. Combined with Theorem 3.19, we find a complete algorithm to compute \( R_u(\text{Gal}^\delta(\mathcal{U})) \). Theorem 3.19 implies among other things that \( R_u(\text{Gal}^\delta(\mathcal{U})) \), \( \text{Gal}^\delta(\mathcal{U}) \), and \( \text{Gal}(\mathcal{U}) \) are Zariski dense in \( \text{Gal}(\mathcal{U}) \) (resp. \( \text{Gal}(\mathcal{L}) \)). It might happen that \( R_u(\text{Gal}^\delta(\mathcal{U})) \subseteq R_u(\text{Gal}(\mathcal{U})) \) as it is shown in the following example.

**Example 3.20.** Let \( V = \text{span}_k \{x^2, xy, y^2, x' y - xy'\} \subset k[x, y] \), and let us consider the following representation \( \rho : \text{PSL}_2 \to \text{GL}(V) \) (cf. [30, Example 3.7]):

\[
\text{PSL}_2 \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mod \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \mapsto \begin{pmatrix} a^2 & ab & a'b - ab' \\ 2ac & ad + bc & 2(ad' - bc') \\ c^2 & cd & c'd - cd' \end{pmatrix}, \quad (3.12)
\]

Note that \( \overline{\rho}(\text{PSL}_2) = G^3_a \rtimes \text{PSL}_2 \), and we have: \( R_u(\text{PSL}_2) = \{e\} \) but \( R_u(G^3_a \rtimes \text{PSL}_2) = G^3_a \). By [43, Theorem 1.1 and Lemma 2.2], we can construct a \( \delta \)-module \( \mathcal{U} \) such that \( \text{Gal}^\delta(\mathcal{U}) = \text{PSL}_2 \), and \( \rho \) is the representation of \( \text{Gal}^\delta(\mathcal{U}) \) on \( \omega(\mathcal{U}) \) (and so \( \text{Gal}(\mathcal{U}) = G^3_a \rtimes \text{PSL}_2 \)). We can also construct a \( \delta \)-module \( \mathcal{L} \) such that \( \mathcal{U} \) is an extension of \( 1 \) by \( \mathcal{L} \) in the given representation.

**Proposition 3.21.** Let \( G \) be a reductive linear differential algebraic group and \( V \) be a purely non-constant \( G \)-module. Then every \( G \)-invariant \( \delta \)-subgroup \( A \subset V \) is a submodule.

**Proof.** Let us prove that \( A \) is \( k \)-invariant by induction on \( \text{dim} V \). Let \( B \) be minimal among the non-zero \( G \)-invariant \( \delta \)-subgroups of \( V \) that are contained in \( A \), which exists by the Noetherianity of the Kolchin topology. In what follows, we shall prove that \( k B = B \). Assuming this, by the semisimplicity of \( V \), let \( W \subset V \) be a \( G \)-invariant \( k \)-subspace such that \( V = B \oplus W \). Then \( A = B \oplus (W \cap A) \), and \( k(W \cap A) = W \cap A \) by the inductive hypothesis. Therefore, \( kA = A \).
We now show that there exists \( x \in k \setminus C \) such that \( xB = B \). Since \( V \) is purely non-constant, \( kB \) is a non-constant \( G \)-module. By [29, Theorem 3.3], the representation of \( G \) on \( kB \) is polynomial, that is, it extends to the algebraic representation of \( G \). Then it follows from Theorem 2.23 that there exists a diagonalizable element \( g \in G \) with at least one non-constant eigenvalue (cf. [43, proof of Lemma 2.2]). Let \( T_1 < \overline{T} \) be a \( \delta \)-subgroup generated by \( g \). Then, for some positive integer \( r \), \( g^r \in T_1 =: T \). Since \( C \) is algebraically closed, the eigenvalues of \( g^r \) are not all constant on \( V \). The Zariski closure \( \overline{T} \) is an algebraic torus, that is, there is an algebraic isomorphism \( \alpha : (k^*)^r \to \overline{T} \).

By [6, Proposition 31], \( T \) contains \( T_C := \alpha ((C^*)^r) \). Since the action of \( \overline{T} \) on \( kB \) is completely reducible, we have \( kB = \oplus V_i \), the sum over characters of \( \overline{T} \). Note that each \( V_i \) is also a weight space for \( T_C \) and \( \chi \) is uniquely determined by its restriction onto \( T_C \). Since \( B \) is a \( C \)-linear space [6, Proposition 11], it is a direct sum of weight spaces \( V_i \cap B \) for \( T_C \). By construction, these are weight spaces for \( \overline{T} \), hence for \( T \), too. Since there is a non-constant weight of \( T \ni g \) on \( kB \), there exist \( 0 \neq b \in B \) and \( x \in k \setminus C \) such that \( gb = xb \). We fix such an \( x \). Due to the \( G \)-invariance of \( xb \), we obtain that \( B \cap xB \) is a \( G \)-invariant nonzero \( \delta \)-subgroup of \( B \). Since \( B \) is minimal, \( xB = B \).

On the one hand, the set \( S := \{ a \in k \mid aB < B \} \) is a \( k \)-subalgebra of \( k \). On the other hand,

\[
S = \bigcap_{b \in B} q_b^{-1}(B), \quad q_b : k^* \to V, \quad t \mapsto tb,
\]

is a \( \delta \)-subgroup of \( k \). Hence, by [26, Theorem II.6.3, page 97], \( S = C \) or \( k \). Since \( S \ni x \), \( S = k \). \( \square \)

**Proof of Theorem 3.19.** By Theorem 3.13, \( R_u(Gal^\delta (\mathcal{U})) \) is a \( \delta \)-closed subgroup \( W \) of \( \omega(L) \) stable under the action of \( Gal^\delta (L) \). Proposition 3.21 shows that \( W \) is a \( k \)-vector space and thereby a \( Gal^\delta (\mathcal{L}) \)-module. By \( \delta \)-Tannakian equivalence, \( W = \omega(W) \) for some \( \delta \)-submodule \( W < L < \mathcal{U} \). Thus, it remains to prove that \( W \) is the smallest \( \delta \)-submodule \( \mathcal{L}_0 \) of \( \mathcal{L} \) such that such that the parameterized Galois group of \( \mathcal{U}/\mathcal{L}_0 \) is reductive.

Let us show that the set \( V \) of subobjects \( W \) of \( \mathcal{L} \) such that \( R_u(Gal^\delta (\mathcal{U}/W)) = \{1\} \) admits the smallest subobject with respect to the inclusion. It is enough to prove that, if \( V_1 \) and \( V_2 \) are in \( V \), their intersection \( W \) lies in \( V \). Denote by \( G, G_1, \) and \( G_2 \) the parameterized Galois groups of \( \mathcal{U}/W, \mathcal{U}/V_1, \) and \( \mathcal{U}/V_2 \), respectively. The quotient maps \( \mathcal{U}/W \to \mathcal{U}/V_i \) give rise to homomorphisms \( q_i : G \to G_i, i = 1, 2. \) Since \( G_i \) are reductive, \( R_u(G) < \ker q_i. \) Hence, it suffices to show that \( \ker q_1 \cap \ker q_2 = \{1\} \). For each \( g \in G \), the condition \( g \in \ker q_i \) means that \( g(u) = u \in \ker(V_i) \) for all \( u \in \ker(\mathcal{U}). \) Therefore, every element of \( \ker q_1 \cap \ker q_2 \) acts trivially on \( \omega(\mathcal{U})/\omega(W) \).

As in the notation of Lemma 3.6, let \( s \) be a \( k \)-linear section of

\[
0 \to \omega(\mathcal{L}) \to \omega(\mathcal{U}) \to k \to 0
\]

and let \( \xi_\mathcal{U} \) be its associated cocycle. By Lemma 3.6 and Proposition 3.21, the cocycle \( \xi_\mathcal{U} \) identifies \( R_u(Gal^\delta (\mathcal{U})) \) with a \( k \)-vector subgroup \( W = \omega(W) \) of \( \omega(\mathcal{L}) \) for some \( \delta \)-submodule \( W < \mathcal{U} \). To conclude the proof, we have to show that \( W = \omega(\mathcal{L}_0) \).

It follows from the definition of \( \xi \) that the following diagram is commutative:

\[
\begin{array}{ccc}
Gal^\delta (\mathcal{U}) & \xrightarrow{\xi_\mathcal{U}} & \omega(\mathcal{L}) \\
\downarrow{\rho} & & \downarrow{\rho} \\
Gal^\delta (\mathcal{U}/W) & \xrightarrow{\xi_{\mathcal{U}/W}} & \omega(\mathcal{L}/W)
\end{array}
\]

where the vertical arrows are induced by the quotient maps. By the definition of \( \mathcal{U} \) and exactness of \( \omega \), the composition \( \beta \xi_{\mathcal{U}/W} \) vanishes on \( R_u(Gal^\delta (\mathcal{U}/W)) \). Since \( \omega(\mathcal{U}/W) \) is a faithful \( Gal^\delta (\mathcal{U}/W) \)-module and \( \omega(\mathcal{L}/W) \) has no trivial \( Gal^\delta (\mathcal{L}/W) \)-module by assumption, and therefore no trivial
Gal^\delta(\mathcal{U}/W)-module, Propositions 3.22 and 3.23 below show that

$$R_u(\text{Gal}^\delta(\mathcal{U}/W)) = \rho(R_u(\text{Gal}^\delta(\mathcal{U}))).$$

Since \( \zeta \) is injective on the unipotent radical, we conclude that the linear differential algebraic group \( \text{Gal}^\delta(\mathcal{U}/W) \) is reductive. Hence, \( W \rightarrow \hat{\mathcal{Z}}_0 \). Now, if we replace \( W \) with a \( \delta \)-submodule \( V \subset \mathcal{U} \) in the above diagram such that \( \text{Gal}^\delta(\mathcal{U}/V) \) is reductive, we will obtain that

$$\omega(V) \supset \tilde{\zeta}(R_u(\text{Gal}^\delta(\mathcal{U}))) = W.$$

Thus, \( \omega(\hat{\mathcal{Z}}_0) \supset W. \)

Recall that unipotent linear differential algebraic groups are connected. (Otherwise they would have unipotent finite quotients, which is impossible.) Hence, for every linear differential algebraic group \( G \), we have \( R_u(G) = R_u(G^\circ) = R_u(G)^\circ \).

**Proposition 3.22.** Let \( \varphi : G \rightarrow H \) be an epimorphism of linear differential algebraic groups. Assume that, for every proper subgroup \( N \subset R_u(H) \) normal in \( H \), the group \( R_u(H/N) \) is not central in \( (H/N)^\circ = H^\circ/N \). Then \( \varphi(R_u(G)) = R_u(H) \).

**Proof.** Set \( N := \varphi(R_u(G)) \subset R_u(H) \). By the surjectivity of \( \varphi \), the group \( N \) is normal in \( H \). Consider the epimorphism of quotients

$$\nu : G/R_u(G) \rightarrow H/N$$

induced by \( \varphi \). The linear differential algebraic group \( \nu^{-1}(R_u(H/N))^\circ \) is normal in the reductive linear differential algebraic group \( (G/R_u(G))^\circ \). Therefore, it is reductive itself. By Theorem 2.23, \( \nu^{-1}(R_u(H/N))^\circ \) is an almost direct product of a \( \delta \)-closed subgroup \( Z \) of a central torus \( T \subset (G/R_u(G))^\circ \) and of quasi-simple linear differential algebraic groups \( H_i \). Since \( H_i \) coincide with their commutator groups, they cannot have unipotent images, hence \( \nu(H_i) = \{e\} \). We conclude that \( \nu(Z) = R_u(H/N) \). Since \( Z \) is central in \( (G/R_u(G))^\circ \) and \( \nu \) is an epimorphism, the group \( \nu(Z) \) is central in \( (H/N)^\circ \). It follows from the assumption that \( N = R_u(H) \). \( \Box \)

**Proposition 3.23.** The assumption on \( H \) in Proposition 3.22 is satisfied if there exists a short exact sequence

$$0 \rightarrow V \rightarrow U \rightarrow 1 \rightarrow 0$$

of \( H^\circ \)-modules, where \( U \) is a faithful \( H^\circ \)-module and \( V \) is a \( H^\circ \)-semisimple module with no non zero trivial \( H^\circ \)-module, i.e., no non zero \( H^\circ \)-submodule \( W \) fixed by \( H^\circ \).

**Remark 3.24.** Note that if the \( H^\circ \)-module \( V \) has no trivial \( H^\circ \)-module, i.e., no \( k \)-finite dimensional vector space fixed the action of \( H^\circ \) then \( V \) has no non zero \( C \)-vector space fixed by the action of \( H^\circ \). In deed let \( f \) be a non zero element of a \( C \)-vector space fixed by \( H^\circ \), then the \( k \)-vector space spanned by \( f \) is fixed by \( H^\circ \).

**Proof.** It suffices to prove the statement for connected \( H \). Let \( N \subset R_u(H) \) be a \( \delta \)-subgroup normal in \( H \) such that \( R_u(H/N) \) is central in \( H/N \). Since we have a commutative diagram

$$\begin{array}{ccc}
H & \rightarrow & H/N \\
\downarrow & & \uparrow \\
R_u(H) & \rightarrow & R_u(H/N),
\end{array}$$

the latter implies that, for all \( g \in R_u(H) \), one has \( hgh^{-1} \in gN \). Let \( u \in U \) be an element whose image in \( 1 \) is non-zero. Moreover, \( R_u(H) \) acts trivially on \( V \) because \( V \) is \( H \)-semi-simple. Thus, the map

$$\zeta : R_u(H) \rightarrow V, \quad g \mapsto gu - u$$
is an $H$-equivariant monomorphism of linear differential algebraic groups (see proofs of Lemmas 3.6 and 3.8), that is, for all $h \in H$ and $g \in R_u(H)$, we have

$$hg u - hu = hg h^{-1} u - u.$$ 

The $\delta$-subgroups $\zeta(R_u(H))$ and $\zeta(N)$ of $V$ are thus stable under the action of $H$. Note that $\zeta(R_u(H))$ and $\zeta(N)$ are $C$-vector space because, as $\delta$-subgroup of $V$, they are zero sets linear homogeneous differential equations over $k$.

Let $n \in N$ be such that $hg h^{-1} = gn$ and $n' \in N$ be such that $gn g^{-1} = n'$. Then

$$h(gu - u) = hg u - hu = gnu - u = n' gu - u + n' u - n' u = n'(gu - u) + n' u - u = gu - u + n' u - u,$$

since $gu - u \in V$ and $R_u(H)$ acts trivially on $V$. Therefore, $H$ acts trivially on $\zeta(R_u(H))/\zeta(N)$. Since $\zeta(R_u(H))$ is $H$-semi-simple as $H$-module over $C$, the $H$-module

$$\zeta(R_u(H))/\zeta(N) \subset \zeta(R_u(H)) \subset V$$

is a $C$-vector space fixed by the action of $H$. This contradicts the assumption on $V$. It follows that $R_u(H) = N$.

3.3.3 A general algorithm

Will will now explain a general algorithm to compute the unipotent radical of a $\delta$-module $\mathcal{U}$, extension of 1 by a completely reducible $\partial$-module $\mathcal{L}$. We recall that $\mathcal{L}$ can be decomposed as the direct sum of a constant $\partial$-module $\mathcal{L}_c$ and a purely non-constant $\partial$-module $\mathcal{L}_{nc}$. Considering the pushouts of the extension $\mathcal{U}$ with respect to the decomposition of $\mathcal{L}$, we find the following two exact sequences of $\partial$-modules:

$$0 \longrightarrow \mathcal{L}_c \longrightarrow \mathcal{U}_c \longrightarrow 1 \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{L}_{nc} \longrightarrow \mathcal{U}_{nc} \longrightarrow 1 \longrightarrow 0.$$

Now, we assume that $K = k(x)$ so that we can use the algorithm contained in [31] to compute $R_u(\text{Gal}^\delta(\mathcal{U}_c))$ and the algorithm of §3.3.2 to compute $R_u(\text{Gal}^\delta(\mathcal{U}_{nc}))$. The quotient map $\mathcal{U} \to \mathcal{U}/\mathcal{U}_c = \mathcal{U}_{nc}$ induces an epimorphism $\alpha : \text{Gal}^\delta(\mathcal{U}) \to \text{Gal}^\delta(\mathcal{U}_{nc})$. Similarly, we find an epimorphism $\beta : \text{Gal}^\delta(\mathcal{U}_c) \to \text{Gal}^\delta(\mathcal{U}_c)$. The following theorem allows us to compare $R_u(\text{Gal}^\delta(\mathcal{U}))$ with the the groups computed above.

**Theorem 3.25.** Let $K = k(x)$, $\mathcal{U}, \mathcal{U}_c, \mathcal{U}_{nc}$ be as above, and $\mathcal{L}$ have no trivial $\partial$-submodules. Then the map

$$\alpha \times \beta : R_u(\text{Gal}^\delta(\mathcal{U})) \to R_u(\text{Gal}^\delta(\mathcal{U}_{nc})) \times R_u(\text{Gal}^\delta(\mathcal{U}_c))$$

is an isomorphism of linear differential algebraic groups.

**Proof.** In the proof below, we use the notion of differential type $\tau(G)$ of a linear differential algebraic group $G$ (see [10, §2.1] and [31, Definition 2.2]). Recall that, in the ordinary case, $\tau$ can only take values $-1, 0$, or $1$. Also recall the following result:

**Lemma 3.26** ([10, Equation (1), p. 195]). Let $G$ be a linear differential algebraic group and $H$ be a normal differential algebraic subgroup of $G$. Then $\tau(G) = \max\{\tau(H), \tau(G/H)\}$. 

24
Let us consider the commutative diagram:

\[
R_u((\text{Gal}^\delta(\mathcal{U}_c))) \xleftarrow{\beta} R_u((\text{Gal}^\delta(\mathcal{U}))) \xrightarrow{\alpha} R_u((\text{Gal}^\delta(\mathcal{U}_{nc})))
\]  

(3.14)

Here, the vertical arrows correspond to embedding via the associated cocycles (see (3.13)). The horizontal arrows of the lower row correspond to natural projections. Note that \( R_u((\text{Gal}^\delta(\mathcal{U}_c))) \), \( R_u((\text{Gal}^\delta(\mathcal{U}))) \), and \( R_u((\text{Gal}^\delta(\mathcal{U}_{nc}))) \) are all abelian groups (see Theorem 3.3). It follows from (3.14) that \( \alpha \times \beta \) is an embedding. Then, by [10, Corollary 2.4] and Lemma 3.26,

\[
\tau(R_u(\text{Gal}^\delta(\mathcal{U}))) \leq \tau(R_u((\text{Gal}^\delta(\mathcal{U}_c))) \times R_u((\text{Gal}^\delta(\mathcal{U}_{nc})))) = \max\{\tau(R_u((\text{Gal}^\delta(\mathcal{U}_c))), \tau(R_u((\text{Gal}^\delta(\mathcal{U}_{nc}))))\}.
\]

Since \( \alpha \) and \( \beta \) are surjective, we find that

\[
\tau\left(R_u(\text{Gal}^\delta(\mathcal{U}))\right) = \max\{\tau(R_u((\text{Gal}^\delta(\mathcal{U}_c))), \tau(R_u((\text{Gal}^\delta(\mathcal{U}_{nc}))))\}.
\]

If \( R_u(\text{Gal}^\delta(\mathcal{U}_{nc})) \neq \{e\} \), it is isomorphic to a nontrivial vector group over \( k \) and its differential type is 1 (see [10, Example 2.9]). Moreover, since the unipotent radicals considered above are \( \delta \)-closed subgroups of vector groups, either they are algebraic groups and their differential type is 1 or they are finite-dimensional \( C \)-vector spaces and their differential type is 0. If \( R_u(\text{Gal}^\delta(\mathcal{U}_{nc})) \) is trivial, we have nothing to prove. Thus, we assume that \( R_u(\text{Gal}^\delta(\mathcal{U}_{nc})) \neq \{e\} \) and that its differential type is 1. By the discussion above, we can also assume that \( \tau(R_u(\text{Gal}^\delta(\mathcal{U}))) = 1 \).

Since \( \mathcal{L} \) has no trivial \( \partial \)-modules, the same holds for \( \mathcal{L}_c \) and \( \mathcal{L}_{nc} \). Now, by Propositions 3.22 and 3.23, \( \alpha \) and \( \beta \) are surjective. Let \( R_0 \leq R_u(\text{Gal}^\delta(\mathcal{U})) \) stand for the strong identity component of \( R_u(\text{Gal}^\delta(\mathcal{U})) \) ([10, Definition 2.6]). Since \( R_u(\text{Gal}^\delta(\mathcal{U}_{nc})) \) is algebraic by Theorem 3.19, it is strongly connected by [10, Lemma 2.8 and Example 2.9]. We have \( \alpha(R_0) = R_u(\text{Gal}^\delta(\mathcal{U}_{nc})) \) (indeed, otherwise \( \alpha(R_0) \subsetneq R_u(\text{Gal}^\delta(\mathcal{U}_{nc})) \)). By definition of the strong identity component, we find that \( \tau(R_u(\text{Gal}^\delta(\mathcal{U}))) / R_0 < 1 \). However, \( \tau(R_u(\text{Gal}^\delta(\mathcal{U}_{nc}))) / \alpha(R_0) = 1 \), because \( R_u(\text{Gal}^\delta(\mathcal{U}_{nc})) \) is strongly connected. Then, we find a surjective map

\[
R_u(\text{Gal}^\delta(\mathcal{U}))/R_0 \rightarrow R_u(G_{nc})/\alpha(R_0)
\]

of a linear differential algebraic group of differential type smaller than 1 onto a linear differential algebraic group of differential type 1, which is impossible. Hence, the group product map

\[
R_0 \times \ker \alpha \rightarrow R_u(\text{Gal}^\delta(\mathcal{U})), \quad (r_0, x) \mapsto r_0 x
\]

is onto. To finish the proof, it suffices to show that \( \beta(\ker \alpha) = R_u(\text{Gal}^\delta(\mathcal{U}_c)) \). If \( \beta(R_0) \neq \{e\} \), it is strongly connected and \( \tau(\beta(R_0)) = \tau(R_0) = 1 \). Since \( \tau(R_u(\text{Gal}^\delta(\mathcal{U}_{nc}))) = 0 \) (see [31, Theorem 2.13]), we have \( \beta(R_0) = \{e\} \) (by Lemma 3.26). Then, \( \beta(\ker \alpha) = R_u(\text{Gal}^\delta(\mathcal{U}_{nc})) \).

\[
\square
\]

4 Criteria of hypertranscendance

We start with a new result in the representation theory of quasi-simple and reductive linear differential algebraic groups, which we further use for our hypertranscendence criterion.
4.1 Extensions of the trivial representation

Let $(k, \delta)$ be a $\delta$-closed field of characteristic 0 and let $C := k^\delta$ be its field of $\delta$-constants. Let $G \subset \text{GL}_n$ be a linear differential algebraic group defined over $k$ and $g \subset k^{n \times n}$ be its Lie algebra [6]. For a $G$-module $W$, we denote the corresponding $g$-module by $\mathfrak{W}$.

**Proposition 4.1.** $W$ is semisimple (equivalently, completely reducible) if and only if so is $\mathfrak{W}$.

**Proof.** By [6, Proposition 30], it suffices to consider the case of a faithful $G$-module $W$. We can then identify $G$ with a Kolchin-closed subgroup of $\text{GL}(W)$. So, $g \subset \text{End}(W)$. By [6, Proposition 20], the $k$-span of $g$ is the Lie algebra of the Zariski closure $\overline{G} \subset \text{GL}(W)$. The proof now follows from the corresponding result for algebraic groups and their Lie algebras [46, Theorem 12.4, p. 97].

Let us call $W$ an adjoint $G$-module if the representation $G \to \text{GL}(W)$ is equivalent to the adjoint representation $\text{Ad} : G \to \text{GL}(g)$ [6, §3.3]. Similarly, $\mathfrak{W}$ is called an adjoint $g$-module if it is equivalent to $g$ with the action on itself given by the Lie bracket.

**Theorem 4.2.** Let

$$0 \to V \to W \to 1 \to 0$$

be an exact sequence of $G$-modules. Assume that $G$ is quasi-simple and that the $G$-module $V$ is simple. If $V$ is not adjoint, then sequence (4.1) of $G$-modules splits.

**Proof.** Suppose $V$ is not adjoint. By Proposition 4.1, it suffices to show that the $g$-module $\mathfrak{W}$ splits. It follows from the classification of simple differential Lie algebras [9, Theorem 19] that $g$ is defined over $C$ and there is a $g(C)$-invariant $C$-vector space $V(C) \subset V$ such that $V \cong k \otimes_C V(C)$ as a $g(C)$-module. In particular, $V(C)$ is not an adjoint $g(C)$-module. Indeed, otherwise $V$ would be an adjoint $k g(C) = g$-module.

As usual, a $(C$-linear) 1-cocycle $\varphi : g \to V$ corresponds to the representation $g \to \text{End}(W)$. Hence, for all $\xi, \eta \in g$,

$$\varphi(\lfloor \xi, \eta \rfloor) = \xi \varphi(\eta) - \eta \varphi(\xi).$$

Since $g(C)$ is a simple Lie algebra, $W$ splits as a $k g(C)$-module. In other words, replacing $\varphi$ by an equivalent cocycle, we have $\varphi(g(C)) = 0$. Hence, $\varphi$ is $g(C)$-equivariant: if $\xi \in g(C)$, then

$$\varphi(\lfloor \xi, \eta \rfloor) = \xi \varphi(\eta).$$

Since $V(C)$ is not adjoint, there are no isomorphic non-trivial simple $g(C)$-submodules in $g$ and $V$. Hence, $\varphi$ vanishes on $g$, and $\mathfrak{W}$ splits.

**Corollary 4.3.** Assume that $G \subset \text{GL}_n$ is a reductive linear differential algebraic group defined over $C$. Let $V$ be a semisimple $G$-module defined over $C$. Let

$$0 \to V \to W \to 1 \to 0$$

be an exact sequence of $G$-modules. If the $G$-modules $g$ and $V$ do not contain isomorphic simple $G$-submodules, then (4.2) splits.

**Proof.** This follows from the proof of Theorem 4.2. 

26
4.2 A handy criterion for hypertranscendence

Let $\Delta = \{\partial, \delta\}$ be a set of two derivations. Let $K$ be a $\Delta$-field with $k := K^\partial$ a $\delta$-closed field. From the results of the previous sections, we obtain the following criterion for the hypertranscendence of the solutions of $L(y) = b$ for irreducible $L \in K[\partial]$.

**Theorem 4.4.** Let $L \in K[\partial]$ be an irreducible $\partial$-operator such that $\text{Gal}(L)$ is a quasi-simple linear algebraic group and $n := \text{ord } L \neq \dim \text{Gal}(L) =: m$. Then, for all $b \in K^*$ and $\Delta$-field extensions $F$ of $K$ with $F^\partial = k$ and containing $z$, a solution of $L(y) = b$ and $u_1, \ldots, u_n$, $K$-linearly independent solutions of $L(y) = 0$, we have:

- the linear differential system $\partial(B) - \delta(A_L) = A_L B - BA_L$ has no solution $B \in K^{n \times n}$, where $A_L$ denotes the companion matrix of $L$ and
- the linear differential equation $L(y) = b$ has no solutions in $K$
if and only if

- the functions $v_1, \ldots, v_m, z, \ldots, \partial^{n-1}z$ and all their derivatives with respect to $\delta$ are algebraically independent over $K$, where $\{v_1, \ldots, v_m\} \subseteq \{u_1, \ldots, \partial^{-1}u_1, \ldots, u_n, \ldots, \partial^{-1}u_n\}$ is a maximal algebraically independent over $K$ subset.

**Example 4.5.** If $L \in K[\partial]$ and $\text{Gal}(L) = \text{SL}_n$, where $n = \text{ord } L \geq 2$, then $L$ is irreducible and $\dim L \neq \dim \text{Gal}(L) = n^2 - 1$. In this situation, in Theorem 4.4, we can take $\{v_1, \ldots, v_m\} = \{u_1, \ldots, \partial^{-n-1}u_1, \ldots, u_n, \ldots, \partial^{-n-2}u_n\}$.

**Proof.** Let $\mathcal{L}$ (respectively, $\mathcal{V}$) be $\partial$-module associated to $L$ (respectively, to $(\partial - \partial(b)/b)L$). Since the $\Delta$-field $K_{\mathcal{V}}$ generated by $u_1, \ldots, u_n, z$ in $F$ is a PPV-extension for $\mathcal{V}$ over $K$, the differential transcendence degree of $K_{\mathcal{V}}$ over $K$ equals the differential dimension of $\text{Gal}^\partial(\mathcal{V})$. Since $\mathcal{L}$ corresponds to the differential system $\partial Y = A_L Y$, Proposition 2.46 together with Theorem 2.23 imply that the first hypothesis is equivalent to $\text{Gal}^\partial(\mathcal{L}) = \text{Gal}(\mathcal{L})$.

Since $L$ is irreducible, there is no nontrivial $\partial$-submodule $\mathcal{N}$ of $\mathcal{L}$ such that the representation of $\text{Gal}^\partial(\mathcal{L})$ on $\omega(\mathcal{N})$ is conjugate to constants, that is, $\mathcal{L}$ is purely non-constant. By Theorem 3.19, $R_n(\text{Gal}^\partial(\mathcal{V})) = \omega(\mathcal{\mathcal{L}}_0)$, where $\mathcal{\mathcal{L}}_0$ is the smallest $\partial$-submodule of $\mathcal{L}$ such that $\text{Gal}^\partial(\mathcal{V} \cap \mathcal{L}_0)$ is reductive. Since $\mathcal{L}$ is irreducible, either $\mathcal{L}_0$ is trivial or $\mathcal{\mathcal{L}}_0 = \mathcal{L}$. The module $\mathcal{\mathcal{L}}_0$ is trivial if and only if $R_n(\text{Gal}^\partial(\mathcal{V})) = k$ if and only if $\omega(\mathcal{V})$ is a $\text{Gal}^\partial(\mathcal{L})$-module. Since $\dim_k \omega(\mathcal{L}) = n$, the $\text{Gal}^\partial(\mathcal{L})$-module $\omega(\mathcal{L})$ is not adjoint. Hence, by the above and Theorem 4.2, we find that $\mathcal{L}_0$ is trivial if and only if the sequence of $\text{Gal}^\partial(\mathcal{L})$-modules

$$0 \to \omega(\mathcal{L}) \to \omega(\mathcal{V}) \to k \to 0,$$

splits, which is equivalent to the existence of a solution in $K$ of the equation $L(y) = b$, in contradiction with the second hypothesis. Therefore, we find that the second hypothesis is equivalent to $R_n(\text{Gal}(\mathcal{V})) = (k^n, +)$, that is, the vector group $G_n^\partial$ and $\text{Gal}^\partial(\mathcal{V}) = G_n^\partial \times \text{Gal}(\mathcal{L})$. The latter is equivalent to $v_1, \ldots, v_m, z, \ldots, \partial^{n-1}z$ being a differential transcendence basis of $K_{\mathcal{V}}$ over $K$. \qed

**Remark 4.6.** The second hypothesis is equivalent to the fact that the parameterized Galois group of $\mathcal{L}$ is not conjugate to constants. For $K$ a computable field, this condition can be tested through various algorithms that find rational solutions (see, for instance, [3]). However, one can sometimes easily prove the non-integrability of the system by taking a close look at the topological generators of the parameterized Galois group such as the monodromy or the Stokes matrices. This is the strategy employed in Lemma 4.7.
4.3 The Lommel differential equation

We apply Theorem 4.4 to the differential Lommel equation, which is a non homogeneous Bessel equation

\[
\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{\alpha^2}{x^2}\right) y = x^{\mu-1},
\]

(4.3)

which depends on two parameters, \(\alpha, \mu \in \mathbb{C}\). We want to study the differential dependencies of the solutions of (4.3) with respect to the parameter \(\alpha\). For this, we consider \(\alpha\) as a new variable, transcendental over \(\mathbb{C}\) and suppose that \(\mu \in \mathbb{Z}\). We endow the field \(\mathbb{C}(\alpha, x)\) with the derivations \(\delta := \frac{\partial}{\partial \alpha}\) and \(\partial := \frac{\partial}{\partial x}\). Now, let \(\mathcal{L}\) be a \(\partial\)-module over \(\mathbb{C}\). Since the \(\Delta\)-fields \(\mathcal{L}\) and \(\mathbb{C}(\alpha, x)\) are linearly disjoints over \(\mathbb{C}(\alpha)\), their compositum \(K := \mathcal{L}(x)\) satisfies \(K^3 = \mathcal{L}\).

Let \(\mathcal{L}\) be a \(\partial\)-module over \(\mathcal{K}\) associated to the Bessel differential equation

\[
L(y) = \frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{\alpha^2}{x^2}\right) y = 0.
\]

(4.4)

Let \(\mathcal{U}\) be a \(\partial\)-module over \(K\) associated to the Lommel differential equation. We have:

\[
0 \rightarrow \mathcal{L} \rightarrow \mathcal{U} \rightarrow 1 \rightarrow 0.
\]

(4.5)

Lemma 4.7. The parameterized Galois group of \(\mathcal{L}\) is \(SL_2\).

Proof. The usual Galois group of \(\mathcal{L}\) is known to be \(SL_2\) [25]. By [9], we know that either \(\text{Gal}^{\delta}(\mathcal{U}) = SL_2\) or \(\text{Gal}^{\partial}(\mathcal{L})\) is conjugate to constants inside \(SL_2\). Suppose that we are in the second situation, that is, there exists \(P \in SL_2\) such that

\[
P \text{Gal}^{\delta}(\mathcal{L}) P^{-1} \subset \{M \in SL_2 | \delta(M) = 0\}.
\]

The coefficients of (4.4) lie in \(\mathbb{C}(\alpha, x)\). Moreover, for a fixed specialization of \(\alpha\) in \(\mathbb{C}\), the point zero is a parameterized regular singular point of (4.4) (see [33, Definition 2.3]). Now, if we fix a fundamental solution \(Z_0\) of (4.4) and we follow [33, page 922], we are able to compute the parameterized monodromy matrices of (4.4) around zero. For a suitable choice of \(Z_0\), we find the following parameterized monodromy matrix,

\[
M_0 = \begin{pmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{pmatrix}
\]

where \(\zeta = e^{2i\pi \alpha}\) and \(\bar{\zeta} = e^{-2i\pi \alpha}\) (see [34, page 35]). By [33, Theorem 3.5], \(M_0\) belongs to some conjugate of \(\text{Gal}^{\partial}(\mathcal{L})\). This means that there exists \(Q \in GL_2\) such that \(\delta(QM_0Q^{-1}) = 0\). Since conjugate matrices have the same spectrum and the spectrum of \(M_0\) is not \(\delta\)-constant, we find a contradiction.

Now, we can prove the following proposition.

Proposition 4.8. Let \(J_\alpha(x)\) be the Bessel function of the first kind and let \(Y_\alpha(x)\) be the Bessel function of the second kind. A solution of the Lommel differential equation is the Lommel function \(s_{\mu, \alpha}(x)\), which is defined as follows

\[
s_{\mu, \alpha}(x) = \frac{1}{2\pi} \left[ Y_\alpha(x) \int_0^x x^\mu J_\alpha(x) \, dx - J_\alpha(x) \int_0^x x^\mu Y_\alpha(x) \, dx \right].
\]

The functions, \(J_\alpha(x), Y_\alpha(x), \frac{d}{dx} Y_\alpha(x), \frac{d}{dx} J_\alpha(x), s_{\mu, \alpha}(x)\) and \(\frac{d}{dx}s_{\mu, \alpha}(x)\) and all their derivatives of all order with respect to \(\frac{\partial}{\partial \alpha}\) are algebraically independent over \(\mathbb{C}(\alpha, x)\). Besides, the parameterized Galois group of \(\mathcal{N}\) is isomorphic to the semi-direct product \(G_\alpha^2 \rtimes SL_2\).
Proof. Since \( \text{Gal}^\delta(L) = \text{SL}_2 \), we just need to prove that \( L(y) = x^{\mu-1} \) has no solution \( g \) in \( K \) in order to apply Theorem 4.4 to the Lommel differential equation. Thus, suppose to the contrary that \( L(y) = x^{\mu-1} \) has a rational solution \( g \in k(x) \). Using partial fraction decomposition, one can show that the only potential pole of \( g \) is zero. Then, let us write \( g = \sum_{j=-p}^{n} a_j x^j \) where \( n, p \) are positive integers and the \( a_j \)'s are elements of \( k \). Since \( g \in k \) is impossible, we can assume that \( n \neq -p \). If we compare the coefficients of monomials of same degree on both sides of the equation \( L(g) = x^{\mu-1} \), we find the following relations

\[
a_n = \begin{cases} 
1 & \mu = n + 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
a_{n-1} = 0
\]

\[
(p^2 - \alpha^2)a_{-p} = \begin{cases} 
1 & \mu = -p - 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
((p - 1)^2 - \alpha^2)a_{-p+1} = 0
\]

\[
(j^2 - \alpha^2)a_j + a_{j-2} = 0, \quad j = -p + 2 \ldots n.
\]

Since either \( a_n \) or \( a_{-p} \) must be nonzero, the integer \( \mu \) must be equal to \( n + 1 \) or \( -p - 1 \). If \( \mu = n + 1 \), then \( -p - 1 \neq \mu \), \( a_{n-1} = a_{-p} = a_{-p+1} = 0 \). Using (4.8), we see that the system has no solution. The argument for \( \mu = -p - 1 \) is symmetric.

\[\square\]

References


[29] A. Minchenko, A. Ovchinnikov, and M. F. Singer. Unipotent differential algebraic groups as parameter-


